A Journey Through Exponential Utility Maximization under Multivariate Fake Stationary Affine Volterra Models.

Emmanuel Gnabeyeu[1]

[1]LPSM, Sorbonne Université and Université Paris Cité.

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Multivariate fake stationary affine Volterra processes with convolutive kernel

Fix $T > 0$, $d \in \mathbb{N}$, under the given complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{0 \le t \le T}, \mathbb{P})$, let $V = (V^1, \ldots, V^d)^\top$ be the following $\mathbb{R}_+^d$–valued scaled Volterra square–root process driven by an d-dimensional process $W = (W^1, \ldots, W^d)^\top$:

$$V_t = \varphi(t) V_0 + \int_0^t K(t-s)\big(\mu(s) + DV_s\big)ds + \int_0^t K(t-s)\nu\varsigma(s)\sqrt{\mathrm{diag}(V_s)}dW_s, \quad V_0 \perp\!\!\!\perp W. \tag{1}$$

Here, $K = \mathrm{diag}(K_1, \ldots, K_d)$ is diagonal with scalar kernels $K_i \in L^2([0, T], \mathbb{R})$, $\varphi = \mathrm{diag}(\varphi^1, \ldots, \varphi^d)$, $\nu = \mathrm{diag}(\nu_1, \ldots, \nu_d)$, $\varsigma = \mathrm{diag}(\varsigma^1, \ldots, \varsigma^d)$ with ς^i a (locally) bounded Borel function and $D := -\mathrm{diag}(\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^{d \times d}$. $\mu : \mathbb{R}_+ \to \mathbb{R}^d$.

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Well-Posedness: Unique-in-law positive continuous weak solution (as a scaling limit of a sequence of time-modulated Hawkes processes) see e.g., [Gnabeyeu, Pagès and Rosenbaum, 2026]

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Definition (Fake Stationarity)

The process $(V_t)_{t \ge 0}$ with diffusion coefficient $\sigma(V)$, starting from $V_0 \in L^2(\mathbb{P})$, exhibit a fake stationary regime of type I if:

$$\forall\, t \ge 0, \quad \mathbb{E}\, V_t = \mathbb{E}\, V_0 =: m_0, \quad \operatorname{Var}(V_t) = \operatorname{Var}(V_0) =: v_0 \quad \text{and} \quad \mathbb{E}\,\sigma^2(V_t) = \bar{\sigma}_0^2. \tag{2}$$

Proposition (Fake stationary Volterra process.)

Let $(V_t)_{t\geq 0}$ be a solution to the scaled Volterra equation in its form (1) starting from any random variable $V_0 \in L^2(\Omega, \mathcal{F}, \mathbb{P})$. Then, a necessary and sufficient condition for the relations (2) to be satisfied is that for $i = 1, \ldots, d$

$$\forall\, t \geq 0, \quad \varphi^i(t) = 1 - D_{ii} \int_0^t K_i(t-s) \left(\frac{\mu^i(s)}{D_{ii} m_0^i} - 1 \right) \mathrm{d}s. \tag{3}$$

and each couple $(v_0^i, \varsigma^i(t))$, where $v_0^i = Var(V_0^i)$ must satisfy the functional equation: ($f_{D_{ii}}$ is the resolvent of $D_{ii} K_i$)

$$(E_{D_{ii},c_i}):\ \forall\, t \geq 0,\ c_i D_{ii}^2 \Big(1 - \big(\varphi^i(t) - (f_{D_{ii}} * \varphi^i)_t\big)^2\Big) = \big(f_{D_{ii}}^2 * \varsigma^{i2}\big)(t) \text{ where } c_i = \frac{v_0^i}{\nu_i^2 m_0^i} \text{ i.e. } \varsigma^i := \varsigma^i_{D_{ii},c_i}. \tag{4}$$

Definition (Stabilizer)

We will call the stabilizer (or corrector) of the scaled stochastic Volterra equation (1) the (locally) bounded Borel function $\varsigma = \mathrm{diag}(\varsigma^1, \ldots, \varsigma^d)$ where ς^i is a solution(if any) to the functional equation (E_{D_{ii},c_i}) in (4) for $i = 1, \ldots, d$.

Example (Computing the Stabilizer $\varsigma_{\alpha_i, D_{ii}, c_i}$ for the α_i-fractional kernels $K_{\alpha_i}(t) = \frac{u^{\alpha_i - 1}}{\Gamma(\alpha_i)} \mathbf{1}_{\mathbb{R}}(t)$)

Within the setting $\mu^i = D_{ii} m_0^i \Rightarrow \varphi^i(t) = \varphi^i(0) = 1$ for all $t \geq 0$ and $i = 1, \dots, d$, the functional equation (E_{D_{ii}, c_i}) reads :

$$(E_{D_{ii}, c_i}): \quad \forall\, t \geq 0,\ c_i D_{ii}^2 \Big(1 - \big(1 - \int_0^t f_{D_{ii}}(s) ds\big)^2\Big) = \big(f_{D_{ii}}^2 * \varsigma^{i2}\big)(t). \tag{5}$$

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Regular Variation (Tauberian theorem) on Laplace transforms of (E_{D_{ii},c_i}) suggests to search $\varsigma^2(t)$ as an expansion of the form (*Exponential Power Series Ansatz*): *Set* $a_k = \frac{1}{\Gamma(\alpha_i k+1)}$, $b_k = \frac{1}{\Gamma(\alpha_i(k+1))}$, $k \geq 0$.

$$\varsigma^2_{\alpha_i,D_{ii},c_i}(t) = c_i D_{ii}^{2-\frac{1}{\alpha}i}\varsigma^2_{\alpha_i}(D_{ii}^{\frac{1}{\alpha}i}t) \quad \textit{where} \quad \varsigma^2_{\alpha_i}(t) := 2\,t^{1-\alpha_i}\sum_{k\geq 0}(-1)^k c_k^i t^{\alpha_i k}. \tag{6}$$

with

$$c_0^i = \frac{\Gamma(\alpha_i)^2}{\Gamma(2\alpha_i-1)\Gamma(2-\alpha_i)}, \quad \textit{and for every } k \geq 1,\ c_k \textit{ is defined inductively by:}$$

$$c_k^i = \frac{\Gamma(\alpha_i)^2 B(\alpha_i(k+1),\, 2(1-\alpha_i))}{\Gamma(2(1-\alpha_i))\,\Gamma(2\alpha_i-1)}\left[(a*b)_k - \alpha_i(k+1)\sum_{\ell=1}^{k} B(\alpha_i(\ell+2)-1,\, \alpha_i(k-\ell-1)+2)\,(b^{*2})_\ell\, c_{k-\ell}\right].$$

where for two sequences of real numbers $(u_k)_{k\geq 0}$ *and* $(v_k)_{k\geq 0}$, *the* Cauchy product *is defined as* $(u*v)_k = \sum_{\ell=0}^k u_\ell v_{k-\ell}$

Example: Computing the Stabilizer $\varsigma_{\alpha_i,D_{ii},c_i}$ for the α_i-fractional kernels $K_{\alpha_i}(t) = \frac{u^{\alpha_i-1}}{\Gamma(\alpha_i)}\mathbf{1}_{\mathbb{R}}(t)$:

Proposition (Existence and Properties of the function $\varsigma^2_{\alpha_i,D_{ii},c_i}$ for $\alpha_i \in (\frac{1}{2}, 1)$ [Pagès, 2024])

The convergence radius of the fractional power series (6) *that defines $\varsigma_{\alpha_i,D_{ii},c_i}$ is infinite and $\varsigma_{\alpha_i,D_{ii},c_i}$ is positive on $[0, +\infty]$ so that $\varsigma_{\alpha_i,D_{ii},c_i}$ is well-defined: The stabilizer $\zeta^2_{\alpha_i,D_{ii},c_i}$ exists as an non-negative function, such that:*

1. $\varsigma_{\alpha_i,D_{ii},c_i}(0) = 0$ *and* $\lim_{t\to+\infty} \varsigma_{\alpha_i,D_{ii},c_i}(t) = \frac{\sqrt{c_i}D_{ii}}{\|f_{\alpha_i,D_{ii}}\|_{L^2(Leb_1)}}$.

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Numerical Illustrations: To simulate the volterra process with fractional kernel, we use the K-integrated discrete time Euler scheme on the time grid $t_k=t_k^n=\frac{kT}{n}$, $k=0,\ldots,n$, namely for $i=1,2$, $\overline{V}_0^{i,n}=V^{i,0}$ and for every $k=1,\ldots,n$,

$$\overline{V}_{t_k}^{i,n}=V^{i,0}+\sum_{\ell=1}^{k}\left(\mu_0^i-D_{ii}\overline{V}_{t_{\ell-1}}^{i,n}\right)\underbrace{\int_{t_{\ell-1}}^{t_\ell}K_{\alpha_i}(t_k-s)ds}_{(a)}+\sum_{\ell=1}^{k}\nu_i\varsigma_{\alpha_i,D_{ii},c_i}(t_\ell)\sqrt{\overline{V}_{t_{\ell-1}}^{i,n}}\underbrace{\int_{t_{\ell-1}}^{t_\ell}K_{\alpha_i}(t_k-s)\,dW_s^i}_{(b)} \tag{7}$$

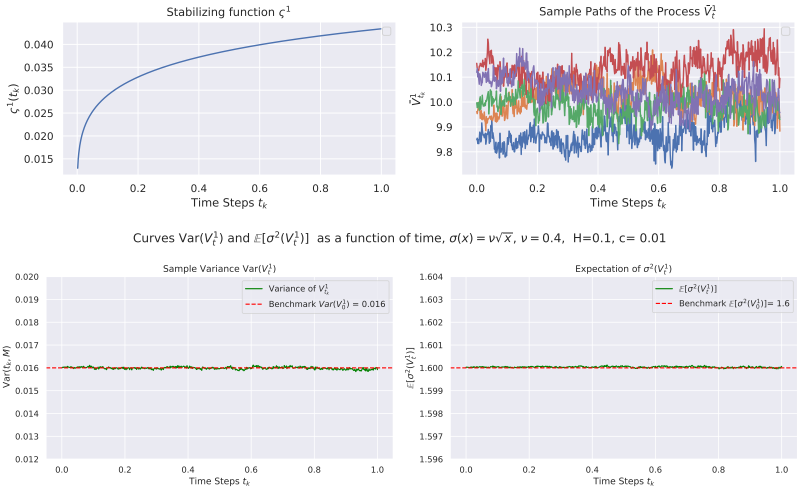

Figure: *Stabilizer, sample paths, empirical variance, and mean squared volatility of V^1 on $[0,1]$ ($H = 0.1$, $c_1 = 0.01$, $n = 600$).*

Curves $\mathrm{Var}(V_t^2)$ and $\mathbb{E}[\sigma^2(V_t^2)]$ as a function of time, $\sigma(x) = \nu\sqrt{x}$, $\nu = 0.32$, H=0.4, c= 0.03

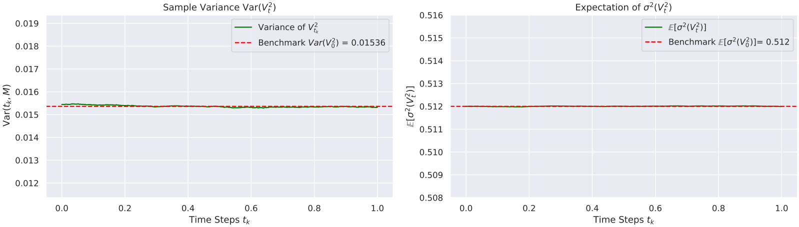

Figure: *Stabilizer, sample paths, empirical variance, and mean squared volatility of* V^2 *on* $[0,1]$ $(H = 0.4,\ c_2 = 0.03,\ n = 600)$.

Literature review on Utility Maximization

Aim and Motivation:

- Maximize an investor's expected utility from terminal wealth with respect to a given utility function (U).
- One of the classical financial economic approaches to understand how volatility affects optimal investment decisions.

The research in applied optimal control with Volterra diffusion models has gained an increasing attention:

- Rough Heston (1D) with power utility: [Bäuerle and Desmettre, 2020]
- Volterra Heston (1D) with power and exponential utilities: [Han and Wong, 2020]
- Multidimensional Volterra Heston models with power utility: [Aichinger and Desmettre, 2021], based on a a Verification argument using calculus of convolutions and resolvents.

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Challenges and Limitations:

- The 1D **Martingale distortion transformation** is no longer available (except in highly degenerate correlation settings).
- Extending the analysis to the multidimensional setting under a non-degenerate correlation structure.

Our contribution: [Gnabeyeu et al., 2026, Gnabeyeu, 2026]

- We solve the exponential utility maximization problem in multivariate fake stationary affine Volterra–Heston models.
- Solution of the problem by BSDEs and Martingale optimality principle, an approach initiated by [Hu et al., 2005]
- We recover the existing results from the literature as special cases.

Consider a financial market on $[0, T]$ with a non–risky asset satisfying an ODE

$$dS_t^0 = S_t^0 r(t)dt \tag{8}$$

and d risky assets (stock or index) whose return vector process $(S_t)_{t\geq 0} = (S_t^1, \ldots, S_t^d)_{t\geq 0}$ is defined via the dynamics given by the vector-stochastic differential equation (SDE):

$$dS_t = \operatorname{diag}(S_t)\big[r(t)\mathbf{1}_d\, dt + \sigma(V_t)\big(dB_t + \lambda_t\, dt\big)\big], \tag{9}$$

driven by a d-dimensional Brownian motion B and $r : \mathbb{R}_+ \to \mathbb{R}$, a time-dependent deterministic short risk-free rate.

- $\sigma(V_\cdot)$: $\mathbb{R}^{d\times d}$-valued continuous stochastic volatility process whose dynamics is driven by (1),
- λ: $\mathbb{R}^d$-valued stochastic market price of risk.

We assume the correlation structure of W with B is given by

$$W = \Sigma B + \sqrt{I - \Sigma^\top \Sigma} B^\perp, \quad \text{with} \quad \Sigma := \operatorname{diag}(\rho_1, \ldots, \rho_d) \tag{10}$$

for some $(\rho_1, \ldots, \rho_d) \in [-1, 1]^d$, and $B^\perp = (B^{\perp,1}, \ldots, B^{\perp,d})^\top$ is an d–dimensional Brownian motion independent of B

- Let π_t denote the vector of the amounts invested in the risky assets S at time t in a self–financing strategy.

Wealth X^π of the portfolio we seek to optimize is given by

$$\mathrm{d}X_t^\pi = X_t^\pi \left(r(t) + \pi_t^\top \sigma(V_t)\lambda_t\right) \mathrm{d}t + \pi_t^\top \sigma(V_t)\, \mathrm{d}B_t. \tag{11}$$

Letting $\alpha_t := \sigma^\top(V_t)\pi_t$ be the investment strategy, the wealth X^α reads:

$$dX_t^\alpha = \left(r(t)X_t^\alpha + \alpha_t^\top \lambda_t\right) dt + \alpha_t^\top dB_t, \quad t \geq 0, \quad X_0^\alpha = x_0 \in \mathbb{R}. \tag{12}$$

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- The set of all admissible investment strategies is denoted as $\mathcal{A}$ and is naturally defined by:

$$\mathcal{A} = \left\{ \begin{array}{l} (\alpha_t)_{t\in[0,T]} \in L^{2,loc}_{\mathbb{F}}([0,T],\mathbb{R}^d) \text{ such that (12) has a solution } X_t^\alpha \text{ for which the family} \\ \left\{\exp\left[-\gamma e^{\int_\tau^T r(u)du} X_\tau^\alpha\right] : \tau \text{ stopping time valued in } [0,T]\right\} \text{ is uniformly integrable.} \end{array} \right\} \tag{13}$$

where given the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, we denote by

$$L^p_{\mathbb{F}}([0,T],\mathbb{R}^d) = \left\{ Y : \Omega \times [0,T] \mapsto \mathbb{R}^d,\ \mathbb{F}-\text{prog. measurable s.t. } \mathbb{E}\Big[\int_0^T |Y_s|^p ds\Big] < \infty \right\}$$

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Find the value function $\mathcal{V}^{\xi}(x_0, V_0)$ for the CARA utility function such that

$$\mathcal{V}^{\xi}(x_0, V_0) = \sup_{\alpha(\cdot)\in\mathcal{A}} \mathbb{E}_{x_0, V_0}\left[U(X_T^{\alpha} - \xi)\right], \quad U(x) := -\frac{1}{\gamma}\exp(-\gamma x), \quad \gamma > 0 \tag{14}$$

where ξ is a $\mathcal{F}_T$-measure random terminal condition and $\gamma > 0$ the risk aversion coefficient of the investor.

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Similar to [Hu et al., 2005], we consider the family of stochastic processes $(J^{\alpha})^{\alpha}$ defined for every $t \in [0, T]$ by

$$J_t^{\alpha} := -\frac{1}{\gamma}\exp(-\gamma x_0 e^{\int_0^T r(s)ds})\exp\left(-\gamma\int_0^t e^{\int_s^T r(u)du}\left(\alpha_s^{\top} dB_s + \alpha_s^{\top}\lambda_s \mathrm{d}s\right) + \gamma Y_t\right) \tag{15}$$

where the pair (Y, Λ) satisfies a Backward SDEs under $\mathbb{P}$ with a driver $f : [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ of the form:

$$\begin{cases} dY_t &= -f(t, Y_t, \Lambda_t)dt + \Lambda_t^{\top} dW_t, \\ Y_T &= \xi =: \int_0^T \sum_{i=1}^d \xi_i V_s^i ds. \end{cases} \tag{16}$$

$\Rightarrow$ We are bound to choose a function f for which J_t^{α} is a supermartingale for all $\alpha \in \mathcal{A}$ and there exists a $\alpha^* \in \mathcal{A}$ such that $J_t^{\alpha^*}$ is a martingale.

Proposition

For any $T > 0$, *an optimal investment strategy* $(\alpha_t^*)_{t\in[0,T]}$ *for the exponential utility maximization problem* (14) *is given by*

$$\alpha_t^* = e^{-\int_t^T r(s)ds}\Big(\frac{\lambda_t}{\gamma} + \Sigma\Lambda_t\Big), \quad 0 \le t \le T. \tag{17}$$

where the pair (Y, Λ) *is a solution (if any) to the quadratic BSDE* (18)

$$\begin{cases} dY_t &= \Big(\frac{1}{2\gamma}\,|\lambda_t + \gamma\Sigma\Lambda_t|^2 - \frac{\gamma}{2}|\Lambda_t|^2\Big)dt + \Lambda_t^\top dW_t. \\ Y_T &= \int_0^T \sum_{i=1}^d \xi_i V_s^i ds. \end{cases} \tag{18}$$

Sketch of Proof: We verify that $\{J_t^\alpha\}_{t\in[0,T]}$, $\alpha \in \mathcal{A}$ satisfies the Martingale optimality principle in the sense that.

1. $J_T^\alpha = U(X_T^\alpha - \xi)$ for all $\alpha \in \mathcal{A}$;
2. J_0^α is a constant, independent of $\alpha \in \mathcal{A}$;
3. J_t^α is a supermartingale for all $\alpha \in \mathcal{A}$, for $\alpha^* \in \mathcal{A}$, J^{α^*} is a martingale (with some additional assumptions).

In this case,

$$\mathbb{E}_{x_0,V_0}\left[U(X_T^\alpha - \xi)\right] = \mathbb{E}_{x_0,V_0}[J_T^\alpha] \le J_0^\alpha = J_0^{\alpha^*} = \mathbb{E}_{x_0,V_0}[J_T^{\alpha^*}] = \mathbb{E}_{x_0,V_0}[U(X_T^{\alpha^*})] = \mathcal{V}^\xi(x_0, V_0). \tag{19}$$

- There is no general existence results covering such quadratic BSDE (unbounded random coefficients)

- We assume that the risk premium λ and the stock volatility σ in (9) are linear in $\sqrt{\text{diag}(V)}$ i.e., are in the form[1]:

$$\sigma(V_t) = \sqrt{\text{diag}(V_t)} \quad \text{and} \quad \lambda_t = \left(\theta_i \sqrt{V_t^i}\right)_{1 \leq i \leq d}, \quad 0 \leq t \leq T. \tag{20}$$

  for some constant $\theta_i \geq 0$.

$\Rightarrow$ This specification is in the spirit of Heston-type model (Fake stationary Volterra Heston volatility).

[1] the BSDE admits a solution characterized by Riccati equations, say now Riccati BSDE

Proposition

Assume that there exists a solution $\psi \in C([0,T],\mathbb{R}^d)$ *to the inhomogeneous Riccati-Volterra equation:*

$$\psi^i(t) = \int_0^t K_i(t-s)\big(\xi_i - \frac{\theta_i^2}{2\gamma} + F_i(T-s,\psi(s))\big)ds, \quad i=1,\dots,d, \tag{21}$$

$$F_i(s,\psi) = -\theta_i\rho_i\nu_i\varsigma^i(s)\psi^i + (D^\top\psi)_i + \gamma\frac{\nu_i^2}{2}(1-\rho_i^2)(\varsigma^i(s)\psi^i)^2. \tag{22}$$

Let Λ *be defined as*

$$\Lambda_t^i := \nu_i\varsigma^i(t)\psi^i(T-t)\sqrt{V_t^i}, \quad i=1,\dots,d, \quad 0\le t\le T. \tag{23}$$

Then, for some $p>1$, (Y,Λ) *is a* $\mathbb{S}_{\mathbb{F}}^p([0,T],\mathbb{R}) \times L_{\mathbb{F}}^2([0,T],\mathbb{R}^d)$*-valued solution to the Riccati BSDE (18) and* Y *reads:*

$$Y_t = \sum_{i=1}^d \int_0^t \xi_i V_s^i ds + \int_t^T \sum_{i=1}^d \big(\xi_i - \frac{\theta_i^2}{2\gamma} + F_i(s,\psi(T-s))\big)g_t^i(s)ds, \tag{24}$$

where the $\mathbb{R}^d$*-valued process* $(g_t(s))_{t\le s}$, $g=(g^1,\dots,g^d)^\top$ *is the adjusted forward variance process defined by:*

$$g_t(s) = \mathbb{E}^{\mathbb{P}}\Big[V_s - \int_t^s K(s-u)DV_u du \,\Big|\, \mathcal{F}_t\Big], \quad \mathbb{P}\text{- a.s., for a.e. } s>t. \tag{25}$$

Remark

Under the condition that $\xi_i - \frac{\theta_i^2}{2\gamma} < 0 \; \forall \, i = 1, \cdots, d.$, *standard arguments provides the existence of a unique global continuous solution* $\psi \in C([0,T], \mathbb{R}^d)$ *to the inhomogeneous Ricatti–Volterra Equation* (21)– (22) *such that* $\psi^i(t) < 0$ *for every* $t > 0$.

Remark

Under the condition that $\xi_i - \frac{\theta_i^2}{2\gamma} < 0 \ \forall\, i = 1, \cdots, d.$, *standard arguments provides the existence of a unique global continuous solution* $\psi \in C([0,T], \mathbb{R}^d)$ *to the inhomogeneous Ricatti–Volterra Equation* (21)– (22) *such that* $\psi^i(t) < 0$ *for every* $t > 0$.

This leads to our main result:

Theorem

Let ψ *be the unique, continuous solution of the inhomogeneous Ricatti-Volterra equation* (21)- (22) *on the time interval* $[0, T]$. *Then, an optimal investment strategy* $(\alpha_t^*)_{t\in[0,T]}$ *for the portfolio exponential utility maximization problem* (14) *is given by*

$$\alpha_t^* = \Big(\frac{1}{\gamma} e^{-\int_t^T r(s)ds} (\theta_i + \gamma \rho_i \nu_i \varsigma^i(t) \psi^i(T-t)) \sqrt{V_t^i}\Big)_{1\le i\le d}, \quad 0 \le t \le T. \tag{26}$$

Moreover α^* *is admissible. The value function defined in* (14) *can be written as*

$$\mathcal{V}^{\xi}(x_0, V_0) = -\frac{1}{\gamma} \exp\Big(-\gamma e^{\int_0^T r(s)ds} x_0 + \gamma \int_0^T \sum_{i=1}^d (\xi_i - \frac{\theta_i^2}{2\gamma} + F_i(s, \psi(T-s))) g_0^i(s) ds\Big). \tag{27}$$

2D fake stationary rough Heston volatility: (d=2) and two differents levels of roughness.

- Recall that $\alpha_t^* = \sigma(V_t)^\top \pi_t^*$ i.e. the optimal strategies $(\alpha_t^*)_{t\in[0,T]}$ are stochastic processes,
- We consider the evolution of the associated deterministic map $t \mapsto \pi_t^*$,

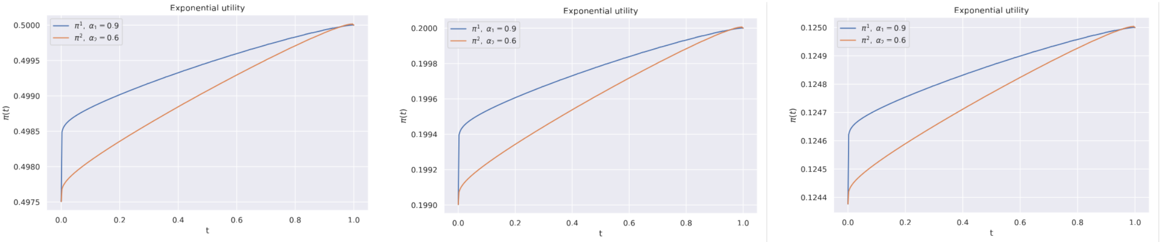
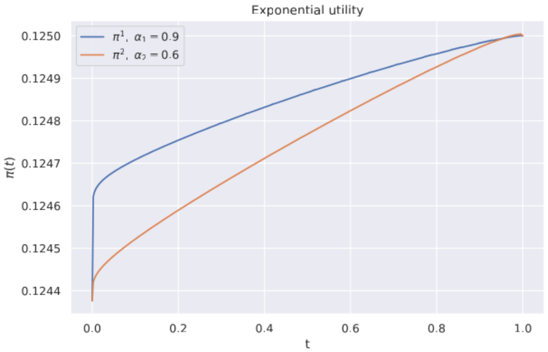


Figure: *Optimal strategy for horizon $T = 1$ and risk aversion levels $\gamma = 0.2$, $\gamma = 0.5$, and $\gamma = 0.8$ (from left to right). Increasing risk aversion leads to a more conservative allocation.*

Our illustrations reveal that:

- the optimal allocation is moderated as the volatility paths become rougher,
- the lower the investor's risk aversion, the higher the optimal allocation.

Numerical Experiments: Impact on the investment horizon



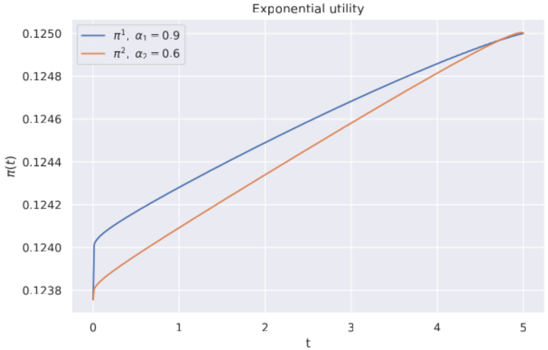


Figure: *Effect of the time horizon T on the optimal strategy for risk aversion $\gamma = 0.8$ ($T = 1$ (left) and $T = 5$ (right)) . The form of the control is consistent across different time scales.*

Regardless of the investment horizon, the stabilizer induces the same functional form for the optimal strategies, reflecting a time-scale invariance property: the functional form of the optimal control is preserved across different investment horizons.

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Thanks For Your Attention! Questions ?