ESILV ENGINEERING SCHOOL DE VINCI PARIS	**Advanced Statistics : Introduction to Bayesian Statistics**	**A4 ESILV** **Tutorial 3**

Exercise 1. BINOMIAL MODEL
Let $X_1, X_2, \ldots, X_n$ be an i.i.d. sample with a Bernoulli $\mathcal{B}(p)$ distribution. Suppose that p has a prior distribution $\beta(a, b)$ with $a, b > 0$.

1. Recall the density of a Beta distribution and give its expectation.
2. What is the prior $\pi(p)$ of p?
3. What is the density of the observations $f(x_1, \ldots, x_n \mid p)$?
4. Calculate the posterior density $\pi(p \mid x_1, \ldots, x_n)$ and deduce that the Beta distribution is conjugate to the Bernoulli distribution.
5. Provide the Bayes estimator of p for the quadratic loss.
6. Propose a credibility interval of level $0 < \alpha < 1$ for the Bayes estimator.

Exercise 2. POISSON MODEL
Let $X_1, X_2, \ldots, X_n$ be an i.i.d. sample from a $P(\Lambda)$ distribution. Suppose that Λ follows a prior $\Gamma(a, b)$ distribution with $a, b > 0$.

1. What is the prior density $\pi(\Lambda)$ of Λ ?
2. What is the density of the observations $f(x_1, \ldots, x_n \mid \Lambda)$?
3. Calculate the posterior density $\pi(\Lambda \mid x_1, \ldots, x_n)$ and deduce that the Gamma distribution is conjugate to the Poisson distribution.
4. Provide the Bayes estimator of Λ for the quadratic loss.
5. Propose a credibility interval of level $0 < \alpha < 1$ for the Bayes estimator.

Exercise 3. EXPONENTIAL MODEL
Let $X_1, X_2, \ldots, X_n$ be an i.i.d. sample from a $\mathcal{E}(\Lambda)$ distribution. Suppose that Λ follows a prior $\Gamma(a, b)$ distribution with $a, b > 0$.

1. What is the prior density $\pi(\Lambda)$ of Λ?
2. What is the density of the observations $f(x_1, \ldots, x_n \mid \Lambda)$?
3. Calculate the posterior density $\pi(\Lambda \mid x_1, \ldots, x_n)$ and deduce that the Gamma distribution is conjugate to the exponential distribution.
4. Provide the Bayes estimator of Λ for the quadratic loss.
5. Propose a credibility interval of level $0 < \alpha < 1$ for the Bayes estimator.

Exercise 4. Bayes 1763

A billiard ball is randomly thrown uniformly on a line of length 1, its random position being denoted θ. Once this is done, a second ball is thrown in the same way n times on the line, and X is the number of times it lands to the left of the first ball, i.e., to the left of θ.

1. What is the prior density $\pi(\theta)$ of θ ?
2. What is the density of the observations $f(x \mid \theta)$?
3. Deduce the posterior density $\pi(\theta \mid x)$ of θ.
4. Adopting the quadratic loss, what is the Bayes estimator of θ?
5. Is the Bayes estimator found biased ? What is its variance ? Deduce its quadratic risk.
6. Provide the estimators by the method of moments and maximum likelihood for X given θ. Are they equal ? Are they biased ? What is their quadratic risk ?
7. Calculate the Bayes risk of the estimators found in the previous questions and compare them. Explain.
8. Propose a credibility interval of level $0 < \alpha < 1$ for the Bayes estimator.

Exercise 5. Bayesian Testing

The following question are independent.

1. Let $X = (X_1, \ldots, X_n) \mid \theta \sim \mathcal{B}(\theta)^{\otimes n}$, with $\theta \in (0,1)$. We aim to test the hypothesis:
$$H_0 : \theta = \frac{1}{2} \quad \text{against} \quad H_1 : \theta \neq \frac{1}{2}.$$
We consider a prior distribution of the form
$$\Pi = \alpha\delta_{1/2} + (1-\alpha)\mathcal{U}([0,1]), \quad \alpha \in (0,1).$$
Show that the Bayes test for H_0 against H_1 for a balanced loss function can be written as
$$\varphi^\star(X) = \mathbf{1}_{\left\{\binom{n}{n\overline{X}_n} \leq \frac{(1-\alpha)2^n}{\alpha(n+1)}\right\}}.$$
2. Let $X = (X_1, \ldots, X_n) \mid \theta \sim \mathcal{N}(\theta, 1)^{\otimes n}$. We consider the hypothesis testing problem:
$$H_0 : \theta = 0 \quad \text{versus} \quad H_1 : \theta \neq 0,$$
and we choose a prior distribution of the form:
$$\Pi = \alpha\delta_0 + (1-\alpha)\mathcal{N}(0, \sigma^2),$$
with $\alpha \in (0,1)$, and $\sigma^2 > 0$. Show that the Bayes test for H_0 versus H_1 for a balanced loss function can be written as
$$\varphi^\star(X) = \mathbf{1}_{\left\{\alpha\sqrt{\sigma^2 n+1} \leq (1-\alpha)\exp\left(\frac{n\overline{X}_n^2}{2(n+\sigma^{-2})}\right)\right\}}.$$
Reminder: For all $a > 0$, $b > 0$, the Beta function is given by:
$$\mathrm{B}(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$