On stationarity of Time-inhomogeneous Affine Volterra process: Finite-Time, Functional Weak Asymptotics and Applications to Rough Heston Model.

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The *short/rough* and *long-term memory* of Volterra Equations

We are interested in the stochastic Volterra integral equation:

$$X_t = X_0\phi(t) + \int_0^t K(t-s)(\theta(s) - \lambda X_s)ds + \int_0^t K(t-s)\sigma(s, X_s)dW_s, \quad X_0 \perp\!\!\!\perp W. \tag{1}$$

where $\theta : [0, T] \to \mathbb{R}$ is a bounded Borel function (hence having a well-defined finite Laplace transform on $\mathbb{R}_+$).

- $\lambda > 0$, $\sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ Lipstchiz continuous function,ϕ a deterministic continuous function.
- $(W_t)_{t\geq 0}$ is a standard Brownian motion, independent of X_0, both defined on a probability space $(\Omega, \mathcal{A}, P)$.
- Let $\mathcal{F}_t \supset \mathcal{F}_{t,X_0,W}$ be a filtration satisfying the usual conditions.
- K is a convolutive kernel, i.e. a kernel $K : \{(s,t) \in \mathbb{R}_+^2 : 0 \leq s < t\} \to \mathbb{R}_+$ satisfying

$$\forall\, s,\ t \geq 0,\ s < t, \quad K(s,t) = K(0, t-s) \tag{2}$$

In this work, we are chiefly interested in the stochastic convolution Volterra integral equation of the form

$$X_t = X_0\phi(t) + \int_0^t K(t-s)(\theta(s) - \lambda X_s)ds + \int_0^t K(t-s)\varsigma(s)\sqrt{\kappa_0 + \kappa_1 X_s}\, dW_s, \quad X_0 \perp\!\!\!\perp W, \tag{3}$$

1. **Resolvent of a Convolutive Kernel**: The λ-resolvent R_λ is defined as the unique solution – if it exists – to

$$\forall t \geq 0, \quad R_\lambda(t) + \lambda \int_0^t K(t-s) R_\lambda(s)\, ds = 1 \quad \Longrightarrow R_\lambda = \sum_{k\geq 0} (-1)^k \lambda^k (1 * K^{*k}) \tag{4}$$

It is used as a generalisation of the classical application of **Itô's Lemma:** to $e^{\lambda t} X_t$.

Summary of Main Tools

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Example (Focus on fractional kernels $K(t) = K_\alpha(t) = \frac{u^{\alpha-1}}{\Gamma(\alpha)}\mathbf{1}_{\mathbb{R}}(t)$)

- $K_\alpha * K_{\alpha'} = K_{\alpha+\alpha'}$ implies $t \geq 0$,

$$R_{\alpha,\lambda}(t) = \sum_{k\geq 0}(-1)^k \frac{\lambda^k t^{\alpha k}}{\Gamma(\alpha k+1)} = E_\alpha(-\lambda t^\alpha), \quad \text{So that} \quad f_{\alpha,\lambda}(t) = -R'_{\alpha,\lambda}(t) = \lambda t^{\alpha-1}\sum_{k\geq 0}(-1)^k\lambda^k \frac{t^{\alpha k}}{\Gamma(\alpha(k+1))}.$$

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2. **Wiener-Hopf Equation:** Let $g : \mathbb{R}_+ \to \mathbb{R}$ be a Borel locally bounded function. Then, the Wiener-Hopf equation

$$\forall t \geq 0, \quad x(t) = g(t) - \lambda \int_0^t K(t-s) x(s)\, ds \tag{5}$$

has a unique solution given by

$$\forall t \geq 0, \quad x(t) = g(t) + \int_0^t R'_\lambda(t-s) g(s)\, ds = g(t) - \int_0^t f_\lambda(t-s) g(s)\, ds \quad \text{where} \quad f_\lambda = -R'_\lambda \tag{6}$$

Main Assumptions

Assumption (We consider the following assumption satisfied by the inhomogeneous Volterra equation (1):)

(i) *Assume that the kernels K, satisfy:*

- *The integrability assumption*

$$(\widehat{\mathcal{K}}^{cont}_{\widehat{\theta}}) \;\; \exists\, \widehat{\kappa} < +\infty,\; \forall \bar{\delta} \in (0,T],\; \widehat{\eta}(\delta) := \sup_{t\in[0,T]} \left[\int_{(t-\bar{\delta})^+}^{t} K(t-u)^2 du \right]^{\frac{1}{2}} \leq \widehat{\kappa}\, \bar{\delta}^{\,\widehat{\theta}} \tag{7}$$

is satisfied for some $\widehat{\theta} \in (0,1]$.

- *The continuity assumption*

$$(\mathcal{K}^{cont}_{\theta}) \;\; \exists\, \kappa < +\infty, ;\, \exists\, \theta \in (0,1] \text{ such that } \forall\, \bar{\delta} \in (0,T),\, \eta(\bar{\delta}) := \sup_{t\in[0,T]} \left[\int_0^t |K((s+\delta)\wedge T) - K(s)|^2 ds \right]^{\frac{1}{2}} \leq \kappa\, \bar{\delta}^{\theta} \tag{8}$$

(ii) *Assume that the drift term b and the diffusion coefficient σ are of linear growth, i.e. there is a constant $C_{b,\sigma} > 0$ such that*

$$|b(t,x)| + |\sigma(t,x)| \leq C_{b,\sigma}(1+|x|), \quad \text{for all } t \in [0,T] \text{ and } x \in \mathbb{R}.$$

(iii) *Assume that the function $\mathbb{R} \ni x \mapsto b(t,x)$ is Lipschitz continuous and $\mathbb{R} \ni x \mapsto \sigma(t,x)$ is Hölder continuous in the space variable uniformly in time of order γ for some $\gamma \in [\frac{1}{2}, 1]$. Hence, there are constants $C_b, C_\sigma > 0$ such that*

$$|\mu(t,x) - \mu(t,y)| \leq C_b|x-y| \quad \text{and} \quad |\sigma(t,x) - \sigma(t,y)| \leq C_\sigma |x-y|^{\gamma} \quad \text{hold for all } t \in [0,T] \text{ and } x,y \in \mathbb{R}.$$

Scaled Volterra Equation: Main Assumption

- **Standing assumption on the kernel K:**

Assumption (λ-resolvent R_λ of the kernel for every $\lambda > 0$:)

$$(\mathcal{K}) \quad \begin{cases} (i) & R_\lambda \quad \mathbb{R}^+ - \textit{differentiable} \ \lim_{t\to+\infty} R_\lambda(t) = a \in [0,1[, \quad R_\lambda(0) = 1, \\ (ii) & f_\lambda \in \mathcal{L}^2_{loc}(\mathbb{R}_+, Leb_1), \textit{Where} \quad f_\lambda = -R'_\lambda \ \textit{for } t > 0, \\ (iii) & \phi \in \mathcal{L}^1_{\mathbb{R}_+}(Leb_1), \ \textit{a continuous function satisfying} \quad \lim_{t\to\infty} \phi(t) = 1 \\ (iv) & \theta \ \textit{is a } C^1\textit{-function such that } \|\theta\|_{sup} < \infty \ \textit{and} \ \lim_{t\to+\infty} \theta(t) = \mu_\infty \in \mathbb{R}, \end{cases} \tag{9}$$

Under these assumptions, f_λ is a 1-sum measure, i.e., $\int_0^{+\infty} f_\lambda(s)\,ds = 1$ and $\lim_{t\to+\infty} \int_0^t f_\lambda(t-s)\theta(s)ds = \mu_\infty$, $\lim_{t\to\infty}(\phi(t) - (f_\lambda * \phi)_t) = a$.

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- **As a consequence of Wiener-Hopf and stochastic/ordinary Fubini's theorems, equation (1) reads:**

$$X_t = X_0\big(\phi(t) - (f_\lambda * \phi)(t)\big) + \frac{1}{\lambda}\int_0^t f_\lambda(t-s)\mu(s)\,ds + \frac{1}{\lambda}\int_0^t f_\lambda(t-s)\sigma(s, X_s)\,dW_s. \tag{10}$$

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Looking for Stationarity !

- Either in the classical sense, where the distribution of the process is invariant under time shifts?
- or in a weaker sense ?

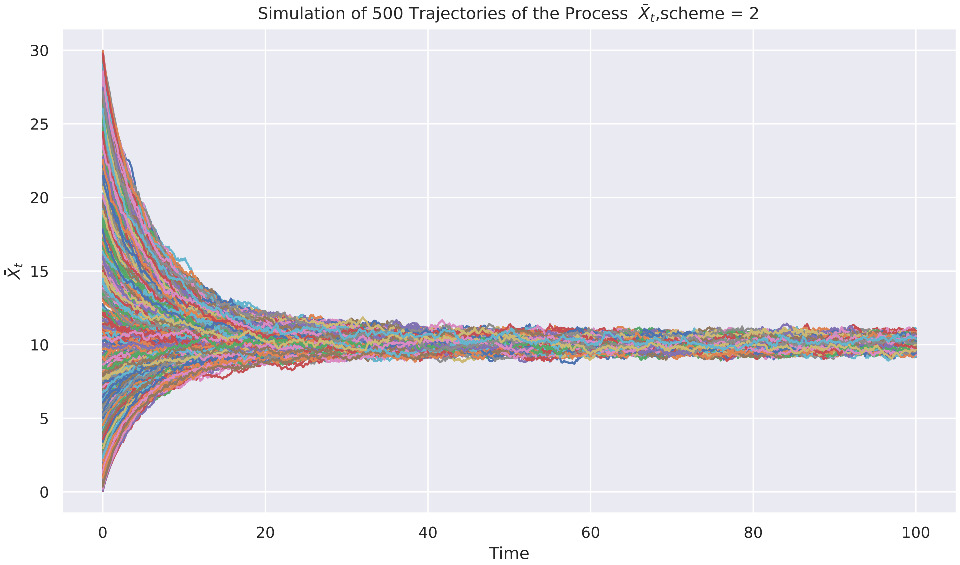

Figure: Confluence from a [0,30]-Uniform Distribution if $\varsigma = \varsigma_{\lambda,c}$ for some $c > 0$ and $\forall\, t \geq 0$, $\quad c\lambda^2\big(1-(\phi(t)-(f_\lambda * \phi)_t)^2\big) = (f_\lambda^2 * \varsigma^2)(t)$

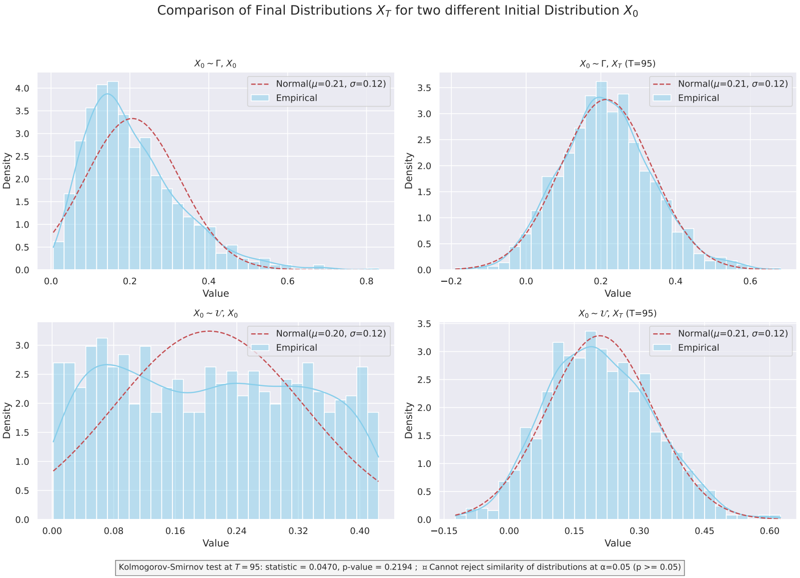

Figure: Histogram of Initial and Final Distributions, $T = 95$, $H = 0.4$, $\mu_0 = 0.0425$, $\lambda = 0.2$, $v_0 = 0.015$. Number of steps: $n = 800$, Simulation size: $M = 100000$. Then Similarity test of Final Distributions

Objectives

1. Long Run behaviour:
   - Existence of limiting distributions ?
   - Existence of associated stationary processes ?
   - Do they depend on the initial state or initial distribution X_0 of the volterra process ?
   - Can we provide a complete characterization of all limiting distributions and explicitly derive the characteristic function of the finite-dimensional distributions of all associated stationary processes ?

     At least, Yes for Inhomogeneous Affine Volterra Equation. (since their (Fourier)-Laplace transform admits an exponential-affine representation)

   - However, their dynamics are analytically intractable (remains an open question): Can we define an alternative dynamics of an SVIE that exhibits some form of stationarity
     (even in a weaker sense) $\Rightarrow$ *"Fake Stationarity Regimes"*?

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2. Finite-time (Fake) Stationarity Regime of Inhomogeneous Affine Volterra Equation:
   - Can we define in a weaker sense, a tractable stationarity regime for affine SVIE ? $\Longrightarrow$ *Fake Stationarity Regimes* à la Pagès.
   - Conditions under which we have a "Fake Stationarity regime" ?
   - Applications to *Stabilized Volterra Heston Model* and its Characterization Function

Lemma (Forward Process)

For all $s, t \in [0, T]$, *we call* $\xi_t(s) := \mathbb{E}[X_s \mid \mathcal{F}_t]$ *the* Forward process *of* X. *Assume that assumptions 2 and* $(\mathcal{K})$ *are satisfied and that* $X = (X_t)_{t\in[0,T]}$ *is a continuous weak solution of (1).* $(X_t)_{t\geq 0}$ *is solution of equation (1) if and only if it is the solution of*

$$X_t = X_0\big(\phi(t) - \int_0^t f_\lambda(t-s)\phi(s)ds\big) + \frac{1}{\lambda}\int_0^t f_\lambda(t-s)\theta(s)ds + \frac{1}{\lambda}\int_0^t f_\lambda(t-s)\sigma(s, X_s)\,dW_s. \tag{11}$$

Then $\xi_t(s)$ *is an* $\mathcal{F}_t$ *-martingale, and for all* $s, t \in [0, T]$ *such that* $t \leq s$, *we have*

$$\mathbb{E}[X_s \mid \mathcal{F}_t] = X_0\phi(s) + \int_0^s K(s-r)\left(\theta(r) - \lambda\mathbb{E}[X_r \mid \mathcal{F}_t]\right)\mathrm{d}r + \int_0^t K(s-r)\sigma(r, X_r)\,\mathrm{d}W_r. \tag{12}$$

Equivalently,

$$\mathbb{E}[X_s \mid \mathcal{F}_t] = X_0(\phi(s) - \int_0^s f_\lambda(s-r)\phi(r)\,dr) + \frac{1}{\lambda}\int_0^s f_\lambda(s-r)\theta(r)\,\mathrm{d}r + \frac{1}{\lambda}\int_0^t f_\lambda(s-r)\sigma(r, X_r)\,\mathrm{d}W_r. \tag{13}$$

Moreover, the forward process $\xi_t(s)$ *satisfies the stochastic differential equation:*

$$\mathrm{d}\xi_t(s) = \frac{1}{\lambda} f_\lambda(s-t)\,\sigma(t, X_t)\,\mathrm{d}W_t, \tag{14}$$

with initial condition (the expected process at future time s*)*

$$\xi_0(s) = \mathbb{E}[X_s] = \mathbb{E}[X_0](\phi(s) - \int_0^s f_\lambda(s-r)\phi(r)\,dr) + \frac{1}{\lambda}\int_0^s f_\lambda(s-r)\theta(r)\,\mathrm{d}r. \tag{15}$$

More generally, We analyse the conditional Laplace functional for $f \in L^1([0,T];\mathbb{R}_-)$:

$$\mathbb{E}\Big[\exp\Big(\int_0^T f(T-s)X_s\,\mathrm{d}s\Big)\,\Big|\,\mathcal{F}_t\Big],\quad t\in[0,T]. \tag{16}$$

To state the main formula in a compact form, let us define and then consider for $f \in L^1(\mathbb{R}_-,\mathbb{R})$, the following time-inhomogeneous Riccati–Volterra equation:

$$\psi(t)=\int_0^t f(s)K(t-s)\,\mathrm{d}s+\int_0^t F(T-s,\psi(s))K(t-s)\,\mathrm{d}s,\quad F(s,\psi)=-\lambda\psi+\frac{\kappa_1}{2}\varsigma^2(s)\psi^2\quad (t,\psi)\in\mathbb{R}_+\times\mathbb{R}. \tag{17}$$

where $\lambda\in\mathbb{R}$, and $\varsigma:\mathbb{R}_+\to\mathbb{R}$ is a given continuous function.

Conditional Fourier-Laplace functional of stabilized affine Volterra processes (1)

More generally, We analyse the conditional Laplace functional for $f \in L^1([0,T];\mathbb{R}_-)$:

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where $\lambda \in \mathbb{R}$, and $\varsigma : \mathbb{R}_+ \to \mathbb{R}$ is a given continuous function.

Proposition (Existence for the time-inhomogeneous Riccati-Volterra equation)

For any function $f \in C([0,T],\mathbb{R}_-) \cap L^1_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}_-)$, the Riccati–Volterra equation (17) admits a unique global solution $\psi = \psi(\cdot,f) \in L^2_{loc}(\mathbb{R}_+;\mathbb{R}_-) \cap C([0,T],\mathbb{R}_-)$, i.e., $\psi(t) \le 0$ for all $t \in [0,T]$. Moreover, we have the following bounds,

1. *If $p \in [1,\infty]$ is such that $K \in L^p_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R})$, then for each $T > 0$, $\|\psi(\cdot,f)\|_{L^p([0,T])} \le \frac{1}{\lambda}\|f\|_{L^1([0,T])}\,\|f_\lambda\|_{L^p([0,T])}$.*
2. **Sobolev-Slobodeckij Regularity of ψ:** *Assume that the kernel $K \in L^2_{loc}(\mathbb{R}_+)$ satisfies the Sobolev-Slobodeckij-type condition $[K]_{\eta,p,T} < \infty$ for some $p \ge 2$, $\eta \in (0,1)$, and $T > 0$. Then the unique solution ψ of (17) belongs to the fractional Sobolev space $W^{\eta,p}([0,T])$, and satisfies the Sobolev-Slobodeckij a priori estimate:*

$$\|\psi(\cdot,f)\|_{W^{\eta,p}([0,T])} \le \|\psi(\cdot,f)\|_{L^p([0,T])} + C[K]_{\eta,p,T}\Big(\|f\|^p_{L^1([0,T])} + \|\psi(\cdot,f)\|_{L^1([0,T])} + \|\psi(\cdot,f)\|^2_{L^2([0,T])}\Big)$$

where the constant $C > 0$ depends only on T, p, λ, κ_1, and the norm of ς in $L^\infty([0,T])$.

Conditional Fourier-Laplace functional of stabilized affine Volterra processes

Theorem

1. *For any function* $f \in C([0,T],\mathbb{R}_-)$*, the Riccati–Volterra equation* (17) *admits a unique global solution* $\psi \in C([0,T],\mathbb{R}_-)$*, i.e.,* $\psi(s) \leq 0$ *for all* $s \in [0,T]$*. Furthermore, the following exponential-affine transform formula holds for the (Fourier–)Laplace transform of* X_T*:*

$$\mathbb{E}\left[\exp\left(\int_0^T f(T-s)X_s\,\mathrm{d}s\right)\Big|\mathcal{F}_t\right] = \exp\left(\int_0^T f(T-s)\xi_t(s)\,\mathrm{d}s + \frac{1}{2}\int_t^T \varsigma^2(s)\sigma^2(\xi_t(s))\,\psi(T-s)^2\,\mathrm{d}s\right), \quad (18)$$

where the process $\xi_t(s)$ *is given by*

$$\xi_t(s) = \xi_0(s) + \frac{1}{\lambda}\int_0^t f_\lambda(s-r)\,\sigma(r,X_r)\,\mathrm{d}W_r, \quad \textit{for } t \leq s,$$
$$\xi_0(s) = \mathbb{E}\left[x_0(s)\right] - \int_0^s f_\lambda(s-r)\mathbb{E}\left[x_0(r)\right]\,dr + \frac{1}{\lambda}\int_0^s f_\lambda(s-r)\theta(r)\,\mathrm{d}r.$$

2. *The inhomogeneous affine Volterra process* (3) *satisfies the exponential-affine transformation formula for the (Fourier-)Laplace transform:*

$$\begin{aligned}\mathbb{E}\left[\exp\left(\int_0^T f(T-s)\,X_s\,ds\right)\right] &= \exp\left(\int_0^T f(T-s)\,\mathbb{E}[X_s]\,ds + \frac{1}{2}\int_0^T \varsigma^2(s)\,\sigma^2(\mathbb{E}[X_s])\,\psi(T-s)^2\,ds\right)\\ &= \exp\left(\int_0^T f(T-s)\,\mathbb{E}[x_0(s)]\,ds + \int_0^T F(s,\psi(T-s))\,\mathbb{E}[x_0(s)]\,ds\right.\\ &\quad\left. + \int_0^T \theta(s)\,\psi(T-s)\,ds + \frac{\kappa_0}{2}\int_0^T \varsigma^2(s)\,\psi^2(T-s)\,ds\right)\end{aligned} \quad (19)$$

Extension to measure convolutions

Lemma (Friesen and Jin)

We work with the subset $\mathcal{M}_{lf}^- \subset \mathcal{M}_{lf}$ *of* $\mathbb{R}_-$*-valued set functions* $\mu \in \mathcal{M}_{lf}$ *negative on* $\mathbb{R}_+$*.*
For each $\mu \in \mathcal{M}_{lf}^-$ *there exists a sequence* $(f_n)_{n\geq 1} \subset L^1_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}_-)$ *such that:*

(i) $\|f_n\|_{L^1([0,T])} \leq |\mu|([0,T])$ *for all* $T > 0$;

(ii) *For each* $T > 0$, $p \geq 1$, *and* $g \in L^p([0,T];\mathbb{R})$ *one has* $g * f_n \to g * \mu$ *in* $L^p([0,T])$;

(iii) *For each* $T > 0$ *and each* $g \in C([0,T];\mathbb{R})$ *with* $g(0) = 0$, $\lim_{n\to\infty} \int_0^t g(t-s)f_n(s)ds = \int_0^t g(t-s)\,\mu(ds)$, $\forall t \in [0,T]$.

Lemma (Friesen and Jin)

We work with the subset $\mathcal{M}_{lf}^- \subset \mathcal{M}_{lf}$ of $\mathbb{R}_-$-valued set functions $\mu \in \mathcal{M}_{lf}$ negative on $\mathbb{R}_+$.
For each $\mu \in \mathcal{M}_{lf}^-$ there exists a sequence $(f_n)_{n\geq 1} \subset L^1_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}_-)$ such that:

(i) $\|f_n\|_{L^1([0,T])} \leq |\mu|([0,T])$ *for all* $T > 0$;

(ii) *For each* $T > 0$, $p \geq 1$, *and* $g \in L^p([0,T];\mathbb{R})$ *one has* $g * f_n \to g * \mu$ *in* $L^p([0,T])$;

(iii) *For each* $T > 0$ *and each* $g \in C([0,T];\mathbb{R})$ *with* $g(0) = 0$, $\lim_{n\to\infty} \int_0^t g(t-s) f_n(s) ds = \int_0^t g(t-s)\,\mu(ds)$, $\forall t \in [0,T]$.

Theorem

Assume $K \in L^2_{loc}(\mathbb{R}_+)$ satisfies the Sobolev-Slobodeckij-type condition $[K]_{\eta,p,T} < \infty$ for some $p \geq 2$, $\eta \in (0,1)$, $T > 0$.

(a) *For each $\mu \in \mathcal{M}_{lf}^-$ there exists a unique solution $\psi = \psi(\cdot,\mu) \in L^2_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}_-)$ satisfying*

$$\psi(t) = \int_0^t K(t-s)\,\mu(ds) + \int_0^t K(t-s)\,F(T-s,\psi(s))\,ds, \quad t \geq 0, \quad F(s,\psi) = -\lambda\psi + \frac{\kappa_1}{2}\varsigma^2(s)\psi^2. \tag{20}$$

where $\lambda \in \mathbb{R}$, and $\varsigma : \mathbb{R}_+ \to \mathbb{R}$ is a given continuous function.

(b) *(L^p-bounds): For each $q \in [1,p]$, $\|\psi(\cdot,\mu)\|_{L^q([0,T])} \leq \frac{1}{\lambda}|\mu|([0,T])\,\|f_\lambda\|_{L^q([0,T])}$. Moreover, there exists $C > 0$ independent of μ such that*

$$\|\psi(\cdot,\mu)\|_{W^{\eta,p}([0,T])} \leq \|\psi(\cdot,\mu)\|_{L^p([0,T])} + C[K]_{\eta,p,T}\left(\|f\|^p_{L^1([0,T])} + \|\psi(\cdot,\mu)\|_{L^1([0,T])} + \|\psi(\cdot,\mu)\|^2_{L^2([0,T])}\right).$$

Remark (On Theorem 7)

Theorem 7 applied for $T = \infty$ *still provides the desired integrability* $\int_0^\infty \left(|\psi(t,\mu)| + |\psi(t,\mu)|^2\right) dt < \infty$. *but with*

$$\psi(t,\mu) = \int_0^t K(t-s)\,\mu(ds) + \int_0^t K(t-s)\,F_\infty(\psi(s,\mu))\,ds, \quad t \geq 0, \quad F(s,\psi) = -\lambda\psi + \frac{\kappa_1}{2}\varsigma^2(s)\psi^2. \tag{21}$$

where F_∞ *is defined as follows:*

$$F_\infty(\psi) := -\lambda\psi + \frac{\kappa_1}{2}\varsigma_\infty^2\psi^2 \quad \text{and} \quad \varsigma_\infty^2 := \lim_{t\to+\infty}\varsigma^2(t) \tag{22}$$

Proposition

Under the same assumptions as in Theorem 7 so that the Riccati–Volterra equation (20) *admits a unique global solution* $\psi = \psi(\cdot,\mu) \in C([0,T],\mathbb{R}_-)$ *for each* $T > 0$, *the following exponential-affine transform formula holds for the measure Fourier–Laplace transform of* X_T*:*

$$\mathbb{E}\left[\exp\left(\int_0^T X_{T-s}\,\mu(ds)\right)\right] = \exp\left(\int_0^T \mathbb{E}[X_{T-s}]\,\mu(ds) + \frac{1}{2}\int_0^T \varsigma^2(s)\,\sigma^2(\mathbb{E}[X_s])\,\psi^2(T-s,\mu)\,ds\right) \tag{23}$$

$$= \exp\left(\int_0^T \mathbb{E}[x_0(T-s)]\,\mu(ds) + \int_0^T F(s,\psi(T-s,\mu))\,\mathbb{E}[x_0(s)]\,ds + \int_0^T \theta(s)\psi(T-s,\mu)\,ds + \frac{\kappa_0}{2}\int_0^T \varsigma^2(s)\psi^2(T-s,\mu)\,ds\right). \tag{24}$$

Existence of limiting distributions (1)

We also assume that the function ς is the solution of the equation:

$$(E_{\lambda,c}): \quad \forall\, t \geq 0, \quad c\lambda^2\left(1-(\phi(t)-(f_\lambda * \phi)_t)^2\right) = (f_\lambda^2 * \varsigma^2)(t) \quad \textit{where} \quad c>0 \quad \textit{and} \quad \varsigma = \varsigma_{\lambda,c}. \tag{25}$$

so that, the corresponding time-inhomogeneous SVIE is refered to as a *Stabilized ihomogeneous Volterra Equation*

Proposition (Let the initial state be $X_0 \in L^2(\mathbb{P})$.)

Let X be the stabilized Volterra Equation given by (3) *and let* $\lambda > 0$, $\mu_\infty \in \mathbb{R}$, *where* $\varsigma = \varsigma_{\lambda,c}$, *assumed to be the unique continuous solution to Equation* (25) *for some* $c > 0$ *(so that* $(E_{\lambda,c})$ *is satisfied). Then the following assertions hold:*

(a) *Suppose that the Riccati-Volterra equation* (17) *has a unique global solution* $\psi \in \mathcal{C}([0,T],\mathbb{C})$ *for each* $T>0$. *Then* $\psi \in L^1(\mathbb{R}_+;\mathbb{R}_-) \cap L^2(\mathbb{R}_+;\mathbb{R}_-)$, *and*

$$\lim_{t\to\infty} \mathbb{E}\left[\exp\left(\int_0^t X_{t-s}\,\mu(ds)\right)\right] = \exp\left[\left(a\bar{x}_0 + (1-a)\frac{\mu_\infty}{\lambda}\right)\mu(\mathbb{R}_+) + \frac{\varsigma_\infty^2}{2}\sigma^2\left(a\bar{x}_0 + (1-a)\frac{\mu_\infty}{\lambda}\right)\int_0^\infty \psi(s,\mu)^2 ds\right] \tag{26}$$

$$= \exp\left[\bar{x}_0\,\mu(\mathbb{R}_+) + \left(\int_0^\infty F_\infty(\psi(s,\mu))\,ds\right)\bar{x}_0 + \left(\int_0^\infty \psi(s)\,ds\right)\mu_\infty + \frac{\kappa_0}{2}\varsigma_\infty^2\int_0^\infty \psi^2(s,\mu)\,ds\right] \tag{27}$$

where F_∞ *is defined as follows:*

$$F_\infty(\psi) := -\lambda\psi + \frac{\kappa_1}{2}\varsigma_\infty^2\psi^2 \quad \textit{and} \quad \varsigma_\infty^2 := \frac{c\lambda^2(1-a^2)}{\|f_\lambda\|^2_{L^2(Leb_1)}} \tag{28}$$

Theorem (Limiting Distribution)

Let X be the stabilized Volterra Equation given by (3) *and let* $\lambda > 0$, $\mu_\infty \in \mathbb{R}$, *where* $\varsigma = \varsigma_{\lambda,c}$, *assumed to be the unique continuous solution to Equation* (25) *for some* $c > 0$ *(so that condition* $(E_{\lambda,c})$ *is satisfied). Let also* $X_0 \in L^2(P)$ *be the initial state. Then the law of the random variable* X_t *converges for* $t \to \infty$ *weakly to a limiting distribution* $\pi_{\bar{x}_0}$, *whose (Fourier)-Laplace transform is for* $u \in \mathbb{R}_-$ *given by*

$$\int_{\mathbb{R}_+} \exp(u\,x)\,\pi_{\bar{x}_0}(dx) = \exp\left[u\bar{x}_0 + \left(\int_0^\infty F_\infty(\psi(s,u\,\delta_0))\,ds\right)\bar{x}_0 + \left(\int_0^\infty \psi(s,u\,\delta_0)\,ds\right)\mu_\infty + \frac{\kappa_0}{2}\varsigma_\infty^2\int_0^\infty \psi^2(s,u\,\delta_0)\,ds\right]$$
$$= \exp\left[u\left(a\bar{x}_0 + (1-a)\frac{\mu_\infty}{\lambda}\right) + \frac{\varsigma_\infty^2}{2}\sigma^2\left(a\bar{x}_0 + (1-a)\frac{\mu_\infty}{\lambda}\right)\int_0^\infty \psi^2(s,u\,\delta_0)ds\right].$$

where F_∞ *is defined as follows:*

$$F_\infty(\psi) := -\lambda\psi + \frac{\kappa_1}{2}\varsigma_\infty^2\psi^2 \quad \text{and} \quad \varsigma_\infty^2 := \frac{c\lambda^2(1-a^2)}{\|f_\lambda\|^2_{L^2(Leb_1)}} \tag{29}$$

Moreover, $\pi_{\bar{x}_0}$ *has finite first moment.*

Long run behavior: functional weak asymptotics (1)

Assumption (Integrability and Uniform Hölder Continuity)

- *Let* λ, $c > 0$. *Assume assumption* $(\mathcal{K})$ *in* (**??**) *is in force and the kernel* K *is such that the derivative* $-f_\lambda$ *of its* λ*-resolvent* $R_{\alpha,\lambda}$ *satisfy* $\exists\,\widehat{\kappa} < +\infty$, $\forall\bar{\delta} > 0$,

$$\max_{i=1,2}\sup_{t\in[0,T]}\left[\int_{(t-\bar{\delta})^+}^{t}|f_\lambda(t-u)|^i du\right]^{1/i} \le \widehat{\kappa}\,\bar{\delta}^{\,\widehat{\theta}} \quad \text{for some } \widehat{\theta}\in(0,1]\,, \quad \exists\vartheta\in(0,1]: \max_{i=1,2}\left[\int_0^{+\infty}|f_\lambda(u+\bar{\delta})-f_\lambda(u)|^i\,du\right]^{\frac{1}{i}} \le C\bar{\delta}^{\vartheta}.$$

- *Assume* $(E_{\lambda,c})$ *in force with a stabilizer* $\varsigma_{\lambda,c}$ *and there exists* $\eta\in(0,1)$ *such that* $[K]_{\eta,2,T} < \infty$ *for each* $T > 0$.
- *Moreover, for some* $\delta > 0$, *for any* $p > 0$ *and* $T > 0$, $t \to x_0(t) = X_0\phi(t)$ *is absolutely continuous,and satisfy*

$$\mathbb{E}\Big(\sup_{t\in[0,T]}|x_0(t)|^p\Big) < +\infty, \quad \mathbb{E}\big[\,|x_0(t')-x_0(t)|^p\,\big] \le C_{T,p}\Big(1+\mathbb{E}\big[\sup_{t\in[0,T]}|x_0(t)|^p\big]\Big)|t'-t|^{\delta p}.$$

Long run behavior: functional weak asymptotics (1)

Assumption (Integrability and Uniform Hölder Continuity)

- *Let* λ, $c > 0$. *Assume assumption* $(\mathcal{K})$ *in* (**??**) *is in force and the kernel* K *is such that the derivative* $-f_\lambda$ *of its* λ*-resolvent* $R_{\alpha,\lambda}$ *satisfy* $\exists\, \widehat{\kappa} < +\infty,\ \forall \bar{\delta} > 0,$

$$\max_{i=1,2} \sup_{t\in[0,T]} \Big[\int_{(t-\bar{\delta})^+}^{t} |f_\lambda(t-u)|^i du\Big]^{1/i} \le \widehat{\kappa}\,\bar{\delta}^{\widehat{\theta}} \quad \text{for some } \widehat{\theta} \in (0,1]\,, \quad \exists \vartheta \in (0,1] : \max_{i=1,2} \Big[\int_0^{+\infty} |f_\lambda(u+\bar{\delta}) - f_\lambda(u)|^i\, du\Big]^{\frac{1}{i}} \le C\bar{\delta}^{\vartheta}.$$

- *Assume* $(E_{\lambda,c})$ *in force with a stabilizer* $\varsigma_{\lambda,c}$ *and there exists* $\eta \in (0,1)$ *such that* $[K]_{\eta,2,T} < \infty$ *for each* $T > 0$.
- *Moreover, for some* $\delta > 0$, *for any* $p > 0$ *and* $T > 0$, $t \to x_0(t) = X_0\phi(t)$ *is absolutely continuous,and satisfy*

$$\mathbb{E}\Big(\sup_{t\in[0,T]} |x_0(t)|^p\Big) < +\infty, \quad \mathbb{E}\big[\,|x_0(t') - x_0(t)|^p\,\big] \le C_{T,p}\Big(1 + \mathbb{E}\big[\sup_{t\in[0,T]} |x_0(t)|^p\big]\Big)|t'-t|^{\delta p}.$$

Theorem (Long run theorem-a: functional weak asymptotics (1))

Let X *be the stabilized Volterra Equation given by* (3) *and* $\lambda > 0$, $\mu_\infty \in \mathbb{R}$. *Then the following assertions hold:*

(a) *The family of shifted processes* $(X_{t+u})_{u\ge 0}$ *is* $\mathcal{C}$*-tight and uniformly square integrable for* $p > 2$ *as* $t \to +\infty$.
There exists a stationary process X^∞ *with a* $\Big(\delta \wedge \vartheta \wedge \frac{\beta-1}{2\beta} - \frac{1}{p} - \eta\Big)$*-Hölder pathwise continuous sample paths for sufficiently small* $\eta > 0$ *such that*

$$(X_{t+u})_{t\ge 0} \Rightarrow (X_t^\infty)_{t\ge 0} \quad \text{weakly in } C(\mathbb{R}_+;\mathbb{R}) \text{ as } u \to \infty.$$

Theorem (Long run theorem b-c: Stationary Process. Here and set $\bar{x}_0 := \mathbb{E}(X_0)$)

(b) *Its first moment and its autocovariance function* $Cov(X^\infty_{t_1}, X^\infty_{t_2}) := C_{f_\lambda}(t_1, t_2)$, *for* $t_1, t_2 \geq 0$, $t_1 \leq t_2$, *are given by*

$$\mathbb{E}[X^\infty_t] = a\bar{x}_0 + (1-a)\frac{\mu_\infty}{\lambda}, \ \mathrm{cov}(X^\infty_{t_2}, X^\infty_{t_1}) = a^2\mathrm{Var}(X_0) + \frac{c(1-a^2)}{\|f_\lambda\|^2_{L^2(Leb_1)}}\sigma^2\left(a\bar{x}_0 + (1-a)\frac{\mu_\infty}{\lambda}\right)\int_0^{+\infty} f_\lambda(t_2 - t_1 + u)f_\lambda(u)du.$$

(c) *The finite dimensional distributions of* X^∞ *are determined by* $(n \in \mathbb{N},\ u_1, \ldots, u_n \in \mathbb{R}_-$ *and* $0 \leq t_1 < \cdots < t_n)$ *:*

$$\mathbb{E}\left[\exp\left(\sum_{i=1}^n u_i X^\infty_{t_i}\right)\right] = \exp\left[\sum_{i=1}^n u_i\left(a\bar{x}_0 + (1-a)\frac{\mu_\infty}{\lambda}\right) + \frac{\varsigma^2_\infty}{2}\sigma^2\left(a\bar{x}_0 + (1-a)\frac{\mu_\infty}{\lambda}\right)\int_0^\infty \psi(s)^2 ds\right].$$

where $\psi(\cdot) = \psi(\cdot, \mu_{t_1,\ldots,t_n})$ *denotes the unique solution of* (21) *in* $\mathbb{R}_+$ *with* $\mu_{t_1,\ldots,t_n}(ds) = \sum_{j=1}^n u_j\,\delta_{t_n - t_j}(ds)$

Long run behavior: Stationary Process (2)

Theorem (Long run theorem b-c: Stationary Process. Here and set $\bar{x}_0 := \mathbb{E}(X_0)$)

(b) *Its first moment and its autocovariance function* $Cov(X^\infty_{t_1}, X^\infty_{t_2}) := C_{f_\lambda}(t_1, t_2)$, *for* $t_1, t_2 \geq 0$, $t_1 \leq t_2$, *are given by*

$$\mathbb{E}[X^\infty_t] = a\bar{x}_0 + (1-a)\frac{\mu_\infty}{\lambda}, \; \mathrm{cov}(X^\infty_{t_2}, X^\infty_{t_1}) = a^2\mathrm{Var}(X_0) + \frac{c(1-a^2)}{\|f_\lambda\|^2_{L^2(Leb_1)}}\sigma^2\left(a\bar{x}_0 + (1-a)\frac{\mu_\infty}{\lambda}\right)\int_0^{+\infty} f_\lambda(t_2 - t_1 + u) f_\lambda(u)du.$$

(c) *The finite dimensional distributions of* X^∞ *are determined by* ($n \in \mathbb{N}$, $u_1, \ldots, u_n \in \mathbb{R}_-$ *and* $0 \leq t_1 < \cdots < t_n$) :

$$\mathbb{E}\left[\exp\left(\sum_{i=1}^n u_i X^\infty_{t_i}\right)\right] = \exp\left[\sum_{i=1}^n u_i\left(a\bar{x}_0 + (1-a)\frac{\mu_\infty}{\lambda}\right) + \frac{\varsigma^2_\infty}{2}\sigma^2\left(a\bar{x}_0 + (1-a)\frac{\mu_\infty}{\lambda}\right)\int_0^\infty \psi(s)^2 ds\right].$$

where $\psi(\cdot) = \psi(\cdot, \mu_{t_1,\ldots,t_n})$ *denotes the unique solution of* (21) *in* $\mathbb{R}_+$ *with* $\mu_{t_1,\ldots,t_n}(ds) = \sum_{j=1}^n u_j\, \delta_{t_n - t_j}(ds)$

Corollary (Let the initial state be $X_0 \in L^2(\mathbb{P})$.)

Let X *be the stabilized Volterra Equation given by* (3) *and let* $\lambda > 0$, $\mu_\infty \in \mathbb{R}$. *Then the following are equivalent:*

(i) *The stationary process* X^∞ *is independent of* $\bar{x}_0$;

(ii) *The limiting distribution* $\pi_{\bar{x}_0}$ *is independent of* $\bar{x}_0$;

(iii) *The function* $\bar{x}_0 \longmapsto \int_{\mathbb{R}_+} x\pi_{\bar{x}_0}(dx)$ *is constant;*

(iv) $a := \lim_{t\to+\infty} R_\lambda(t) = 0$. *In particular, final dist. in the fake stationary regime does not depend on the initial dist.*

Applications: Long-Run Behaviour of the Stabilized Volterra-CIR Process with a Gamma Kernel

$$X_t = X_0\phi(t) + \int_0^t K(t-s)(\theta(s) - \lambda X_s)ds + \nu \int_0^t K(t-s)\varsigma_{\lambda,c}(s)\sqrt{X_s}dW_s, \quad X_0 \perp\!\!\!\perp W \tag{30}$$

$K(t) = K_{\alpha,\rho}(t) = e^{-\rho t}\frac{t^{\alpha-1}}{\Gamma(\alpha)}\mathbf{1}_{\mathbb{R}}(t)$, $\alpha \in (\frac{1}{2}, \frac{3}{2})$, $\lambda, \sigma, b, X_0 \geq 0$, $\beta \in \mathbb{R}$, $(W_t)_{t\geq 0}$ is a 1-D BM and $\varsigma_{\lambda,c}$ solution of $(E_{\lambda,c})$ $\quad f_\lambda \equiv f_{\alpha,\rho,\lambda}$.

Theorem (Let X be a weak solution of the stabilized SVIE given by (30). We have the following claims:)

1. *X_t converges weakly to some limiting distribution $\pi_{\bar{x}_0}$ when $t \to \infty$, and that its characteristic function is given by the expression in Theorem 9 and ψ being determined from the ricatti-volterra equation (21) with $\mu(ds) = u\,\delta_0(ds)$,.*
2. *Moreover, the process $(X_{t+u})_{t\geq 0}$ converges in law to a continuous stationary process $(X_t^\infty)_{t\geq 0}$ when $u \to \infty$. Moreover, the finite dimensional distributions of X^∞ have the characteristic function*

$$\mathbb{E}\left[\exp\left(\sum_{i=1}^n u_i X_{t_i}^\infty\right)\right] = \exp\left[\frac{\rho^\alpha \bar{x}_0 + \mu_\infty}{\rho^\alpha + \lambda}\left(\sum_{i=1}^n u_i + \frac{\varsigma_\infty^2}{2}\nu^2\int_0^\infty \psi(s)^2 ds\right)\right], \quad \varsigma_\infty^2 := \frac{c\lambda^2}{\|f_{\alpha,\rho,\lambda}\|^2_{L^2(\mathrm{Leb}_1)}}\frac{\lambda\,(2\rho^\alpha + \lambda)}{(\rho^\alpha + \lambda)^2}.$$

where $0 \leq t_1 < \cdots < t_n$, $u_1, \ldots, u_n \in \mathbb{R}_-$, and ψ is the unique solution of

$$\psi(t) = \sum_{j=1}^n \mathbf{1}_{\{t > t_n - t_j\}} K_{\alpha,\rho}(t - (t_n - t_j))u_j + \int_0^t K_{\alpha,\rho}(t-s)\left(-\lambda\psi(s) + \varsigma_\infty^2\frac{\nu^2}{2}\psi(s)^2\right)ds.$$

Moreover, the first moment and the autocovariance function of the stationary process satisfy for $t_1, t_2 \geq 0$, $t_1 \leq t_2$

$$\mathbb{E}[X_t^\infty] = \frac{\rho^\alpha \bar{x}_0 + \mu_\infty}{\rho^\alpha + \lambda}, \quad \mathrm{Cov}(X_{t_2}^\infty, X_{t_1}^\infty) = \frac{\rho^{2\alpha}\mathrm{Var}(X_0)}{(\rho^\alpha + \lambda)^2} + \frac{c\lambda\nu^2(2\rho^\alpha + \lambda)(\rho^\alpha \bar{x}_0 + \mu_\infty)}{(\rho^\alpha + \lambda)^3\|f_\lambda\|^2_{L^2(\mathrm{Leb}_1)}} e^{-\rho(t_2 - t_1)}\int_{\mathbb{R}_+} e^{-2\rho u} f_\lambda(t_2 - t_1 + u) f_\lambda(u)\,du.$$

Morally speaking

We have provided

- a complete characterization of all limiting distributions
- each one gives rise to a stationary process for which, we explicitly derive the characteristic function of their finite-dimensional distributions.

However, we do not provide information on the dynamics of the limiting processes as well as the uniqueness and the characterization of the dynamics of the corresponding stationary processes.

- It is for this reason that we develop the notion of fake stationarity regimes, which offer a tractable alternative framework to study short and long-term behaviors in settings where classical stationarity is either unavailable or analytically intractable.

Fake Stationarity of Time-Inhomogeneous Affine Volterra Process: Finite-time.

Definition (Fake Stationary Regime of type I and II [2] G.Pagès 2024)

Let $(X_t)_{t\geq 0}$ be a solution to the scaled Volterra equation in its form (1) starting from any $X_0 \in L^2(\mathbb{P})$. Let $\sigma(t,x) = \varsigma(t)\sigma(x)$ in equation (1), where $\sigma(x) = \sqrt{\kappa_0 + \kappa_1 x}$.

1. The process $(X_t)_{t\geq 0}$ exhibit a fake stationary regime of type I if it has constant mean $\frac{\mu_\infty}{\lambda}$, variance v_0, and $\bar{\sigma}_0^2 = \mathbb{E}[\sigma^2(X_0)] = \mathbb{E}[\sigma^2(X_t)]$ i.e.:

$$\forall\, t \geq 0, \quad \mathbb{E}[X_t] = c^{\mathrm{ste}} = \frac{\mu_\infty}{\lambda}, \quad \mathrm{Var}(X_t) = c^{\mathrm{ste}} = v_0 \geq 0 \quad \text{and} \quad \bar{\sigma}^2(t) := \mathbb{E}[\sigma^2(X_t)] = c^{\mathrm{ste}} := \bar{\sigma}_0^2 \geq 0. \tag{31}$$

2. The process $(X_t)_{t\geq 0}$ exhibit a fake stationary regime of type II if $(X_t)_{t\geq 0}$ has the same marginal distribution, i.e., $X_t \stackrel{d}{=} X_0$ for every $t \geq 0$.

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1. The process $(X_t)_{t\geq 0}$ exhibit a fake stationary regime of type I if it has constant mean $\frac{\mu_\infty}{\lambda}$, variance v_0, and $\bar{\sigma}_0^2 = \mathbb{E}[\sigma^2(X_0)] = \mathbb{E}[\sigma^2(X_t)]$ i.e.:

$$\forall\, t \geq 0, \quad \mathbb{E}[X_t] = c^{\mathrm{ste}} = \frac{\mu_\infty}{\lambda}, \quad \mathrm{Var}(X_t) = c^{\mathrm{ste}} = v_0 \geq 0 \quad \text{and} \quad \bar{\sigma}^2(t) := \mathbb{E}[\sigma^2(X_t)] = c^{\mathrm{ste}} := \bar{\sigma}_0^2 \geq 0. \tag{31}$$

2. The process $(X_t)_{t\geq 0}$ exhibit a fake stationary regime of type II if $(X_t)_{t\geq 0}$ has the same marginal distribution, i.e., $X_t \overset{d}{=} X_0$ for every $t \geq 0$.

Theorem (Time-Dependent Volatility σ. Let $\sigma(t,x) = \varsigma(t)\sigma(x)$ in equation (11))

Assume that $X_0 \in L^2(\mathbb{P})$ with $\mathbb{E}[X_0] = \frac{\mu_\infty}{\lambda}$. Then, a necessary condition for the relations 31 to be satisfied is that:

$$\forall\, t \geq 0,\ \phi(t) = 1 - \lambda \int_0^t K(t-s)\left(\frac{\theta(s)}{\mu_\infty} - 1\right) ds, \text{ so that } X_t = X_0 - \left(X_0 - \frac{\mu_\infty}{\lambda}\right)\int_0^t f_\lambda(t-s)\frac{\theta(s)}{\mu_\infty} ds + \frac{1}{\lambda}\int_0^t f_\lambda(t-s)\varsigma(s)\sigma(X_s)dW_s. \tag{32}$$

and the triplet $(v_0, \bar{\sigma}_0^2, \varsigma(t))$, where $v_0 = Var(X_0)$ and $\bar{\sigma}_0^2 = \mathbb{E}[\sigma^2(X_0)]$, must satisfy the following master equation *:*

$$(E_{\lambda,c}): \quad \forall\, t \geq 0, \quad c\lambda^2\left(1 - (\phi(t) - (f_\lambda * \phi)_t)^2\right) = (f_\lambda^2 * \varsigma^2)(t) \quad \text{where} \quad c = \frac{v_0}{\bar{\sigma}^2} \quad \text{and thus} \quad \varsigma = \varsigma_{\lambda,c}. \tag{33}$$

According to assumption 2 (iii) and the concavity of $x \to x^\gamma$ for all $\gamma \in [0,1]$, $\exists\ [\sigma]^2_{\text{Höl}} > 0$

$$\mathbb{E}\left[|\sigma(X_s) - \sigma(X'_s)|^2\right] \leq [\sigma]^2_{\text{Höl}}\mathbb{E}\left[|X_s - X'_s|^2\right]^\gamma. \tag{34}$$

Proposition (γ-Hölder L^2-Contraction)

Assume assumption (2) (ii) so that equation (34) holds. Assume $f_\lambda \in L^2(\mathbb{R}_+, Leb_1)$, $\sigma(t,x) := \varsigma(t)\sigma(x)$ where $\varsigma = \varsigma_{\lambda,c}$ is a non-negative, continuous and bounded solution to (33) for some fixed $\lambda, c > 0$ (i.e. $E_{\lambda,c}$ is in force). Let $p \geq 2$, for $X_0, X'_0 \in L^p(\mathbb{P})$, we consider the solutions to Volterra equation (1)(or equivalently (11)) denoted $(X_t)_{t\geq 0}$ and $(X'_t)_{t\geq 0}$ starting from X_0 and X'_0 respectively. For $p = 2$. $c \in \left(0, \frac{1}{[\sigma]^2_{Höl}}\right)$, set $\rho := c[\sigma]^2_{Höl}$. Then, one has:

(a) *There exists a continuous non-negative function $\varphi_\infty^{\lambda,c,K,\phi} =: \varphi_\infty : \mathbb{R}^+ \to [0,1]$, such that $\varphi_\infty(0) = 1$, $\frac{a^2}{1-\rho(1-a^2)} \leq \lim_{t\to+\infty} \varphi_\infty(t) \leq \left(\frac{\rho(1-a^2)+\sqrt{\rho^2(1-a^2)+4a^2}}{2}\right)^2$, only depending on λ, c, ϕ, and the kernel K, such that :*

$$\forall t \geq 0,\ \mathbb{E}\left(\left|X_t - X'_t\right|\right)^2 \leq \varphi_\infty(t)\left(1 \vee \mathbb{E}\left(\left|X_0 - X'_0\right|^2\right)\right).$$

In particular, if $a = 0$, then $\lim_{t\to+\infty} \varphi_\infty(t) = \rho^{\frac{1}{1-\gamma}}$.

(b) *This result can be written using the 2-Wasserstein distance between X and X':*

$$\forall t \geq 0,\ W_2([X'_t],[X_t]) \leq \varphi_\infty(t)^{1/2}\left(1 \vee W_2([X'_0],[X_0])\right),$$

(c) *((Locally) Lipschitz L^2-Confluence): In the case $a = 0$,*

Fake stationary regimes I and II of Time-Inhomogeneous Affine Volterra Process: Finite-time.

Let's consider a squared affine diffusion coefficient given by: $\sigma(x) = \sqrt{\kappa_0 + \kappa_1 (x - b)}$, $\kappa_i \geq 0$, $i = 0, 1$, $\kappa_0 \geq \kappa_1 b$. (35)

Proposition (Fake stationary regimes (types I and II) and asymptotics: [3])

Consider a stabilized Volterra Equation with the diffusion coefficient σ given by (35) and let $\lambda > 0$, $\mu_\infty \in \mathbb{R}$, where $\varsigma = \varsigma_{\lambda,c}$, assumed to be the unique continuous solution to Equation $(E_{\lambda,c})$ in (33) for some $c > 0$ Then, let $X_0 \in L^2(P)$ with

$\mathbb{E}[X_0] = \frac{\mu_\infty}{\lambda}$, *and* $Var(X_0) = v_0 = \frac{c\sigma^2(\frac{\mu_\infty}{\lambda})}{1 - c\kappa_2}$:

1. **Case $\kappa_1 = 0$ i.e. $\sigma(x)$ is constant.** *(Volterra Ornstein-Uhlenbeck process) Here, the solution $(X_t)_{t \geq 0}$ has a fake stationary regime of type I with mean $\frac{\mu_\infty}{\lambda}$ and variance v_0.*
   - *This represents a fake stationary regime of type II if $X_0 \sim \nu^* := \mathcal{N}\left(\frac{\mu_\infty}{\lambda}, v_0\right)$, so that $X_t \stackrel{d}{=} X_0 \quad \forall t \geq 0$.*
2. *If $\kappa_0 = 0$ and $b = 0$ (Volterra square-root process.) , then $\sigma(x) = \nu\sqrt{x}$, where $\nu = \sqrt{\kappa_1}$:*
   The resulting Volterra equation has a fake stationary regime of type I, i.e.
   $$\forall t \geq 0, \quad \mathbb{E}[X_t] = \frac{\mu_\infty}{\lambda}, \quad Var(X_t) = v_0 = c\sigma^2(\frac{\mu_\infty}{\lambda}) = c\kappa_1 \frac{\mu_\infty}{\lambda}, \quad \mathbb{E}[\sigma^2(X_t)] = \bar{\sigma}_0^2 = \sigma^2(\frac{\mu_\infty}{\lambda}) = \kappa_1 \frac{\mu_\infty}{\lambda}.$$
3. **Case where σ is not constant and not degenerated (i.e. with $\sigma(\frac{\mu_\infty}{\lambda}) \neq 0$).** *The solution $(X_t)_{t \geq 0}$ to (11) has a fake stationary regime of type I, i.e., for all $t \geq 0$,*
   $\forall t \geq 0, \quad \mathbb{E}[X_t] = \frac{\mu_\infty}{\lambda}, \quad Var(X_t) = v_0 = \frac{c\sigma^2(\frac{\mu_\infty}{\lambda})}{1 - c\kappa_2}$, *and* $\mathbb{E}[\sigma^2(X_t)] = \bar{\sigma}_0^2 = \frac{\sigma^2(\frac{\mu_\infty}{\lambda})}{1 - c\kappa_2}$.

Moreover, if $a = 0$, for any starting value $X_0 \in L^2(P)$, the process X mixes i.e. $\mathbb{E}[X_t] \to \frac{\mu_\infty}{\lambda}$ and $Var(X_t) \to v_0$ as $t \to +\infty$ (L^2 -confluence).

Theorem (Functional weak long-run behaviour.)

Under the same conditions as in Theorem 12, if the solution $(X_t)_{t\geq 0}$ of the volterra equation (3) has a fake stationary regime of type I, starting from a random variable X_0 with mean $\frac{\mu_\infty}{\lambda}$ and variance v_0. Then,

(b) *The identities (26) and (27) become:*

$$\lim_{t\to\infty} \mathbb{E}\left[\exp\left(\int_0^t X_{t-s}\,\mu(ds)\right)\right] = \exp\left[\frac{\mu_\infty}{\lambda}\mu(\mathbb{R}_+) + \frac{\varsigma_\infty^2}{2}\sigma^2\left(\frac{\mu_\infty}{\lambda}\right)\int_0^\infty \psi(s)^2 ds\right] \tag{36}$$

$$= \exp\left[\frac{\mu_\infty}{\lambda}\mu(\mathbb{R}_+) + \left(\int_0^\infty F_\infty(\psi(s))\,ds\right)\frac{\mu_\infty}{\lambda} + \left(\int_0^\infty \psi(s)\,ds\right)\mu_\infty + \frac{\kappa_0}{2}\varsigma_\infty^2\int_0^\infty \psi^2(s)\,ds\right] \tag{37}$$

(b) *The family of shifted processes $X_{t+\cdot}$, $t\geq 0$, is C-tight as $t\to+\infty$ and its (functional) limiting distributions are all L^2-stationary processes with mean $\lim_{t\to\infty}\mathbb{E}[X_t] := \mathbb{E}[X_t^\infty] = \frac{\mu_\infty}{\lambda}$ and covariance function C_∞ given by:*

$$\forall t_1, t_2 \geq 0 \quad \text{with} \quad t_1 \leq t_2 \quad C_\infty(t_1,t_2) = a^2 v_0 + \frac{c(1-a^2)}{\|f_\lambda\|^2_{L^2(Leb_1)}}\sigma^2\left(\frac{\mu_\infty}{\lambda}\right)\int_0^\infty f_\lambda(t_2-t_1+u)f_\lambda(u)\,du. \tag{38}$$

(c) *The finite dimensional distributions of X^∞ are determined by ($n\in\mathbb{N}$, $u_1,\ldots,u_n\in\mathbb{R}_-$ and $0\leq t_1<\cdots<t_n$) :*

$$\mathbb{E}\left[\exp\left(\sum_{i=1}^n u_i X_{t_i}^\infty\right)\right] = \exp\left[\sum_{i=1}^n u_i\left(\frac{\mu_\infty}{\lambda}\right) + \frac{\varsigma_\infty^2}{2}\sigma^2\left(\frac{\mu_\infty}{\lambda}\right)\int_0^\infty \psi(s)^2 ds\right].$$

where $\psi(\cdot) = \psi(\cdot,\mu_{t_1,\ldots,t_n})$ denotes the unique solution of (21) in $\mathbb{R}_+$ with $\mu_{t_1,\ldots,t_n}(ds) = \sum_{j=1}^n u_j\,\delta_{t_n-t_j}(ds)$

Applications: Computing the Stabilizer $\varsigma_{\alpha,\lambda,c}$ for the α-fractional kernels $K_\alpha(t) = \frac{u^{\alpha-1}}{\Gamma(\alpha)}\mathbf{1}_{\mathbb{R}}(t)$:

For the numerical illustration, we consider α-fractional kernels with $\alpha \in \left(\frac{1}{2}, 1\right)$ (corresponding to "rough models") and $\alpha \in \left(1, \frac{3}{2}\right)$ (corresponding to "long memory models"), within the setting where

$$\phi(t) = \phi(0) = 1 \quad \text{for all } t \geq 0 \quad \text{almost surely.}$$

In this case, the equation simplifies in the so-called fake stationarity regime (i.e., $\theta(t) = \theta_0$ and $\sigma(x) = \nu\sqrt{x}$) as follows:

$$X_t = \frac{\theta_0}{\lambda} + \left(X_0 - \frac{\theta_0}{\lambda}\right) R_\lambda(t) + \frac{1}{\lambda}\int_0^t f_{\alpha,\lambda}(t-s)\varsigma(s)\sigma(X_s)dW_s. \tag{39}$$

Regular Variation (Tauberian theorem) on Laplace transforms of the Master Equation suggests to search $\varsigma^2(t)$ as an expansion of the form (*Exponential Power Series Ansatz*): *Set* $a_k = \frac{1}{\Gamma(\alpha k+1)}, b_k = \frac{1}{\Gamma(\alpha(k+1))}, \ k \geq 0.$

$$\varsigma^2_{\alpha,\lambda,c}(t) = c\lambda^{2-\frac{1}{\alpha}}\varsigma^2_\alpha(\lambda^{\frac{1}{\alpha}} t) \quad \textit{where} \quad \varsigma^2_\alpha(t) := 2\, t^{1-\alpha}\sum_{k\geq 0}(-1)^k c_k t^{\alpha k}. \tag{40}$$

with

$$c_0 = \frac{\Gamma(\alpha)^2}{\Gamma(2\alpha-1)\Gamma(2-\alpha)}\frac{\mu(0)}{\mu_\infty}, \quad \textit{and for every } k \geq 1,\ c_k \textit{ is defined inductively by:}$$

$$c_k = \frac{\Gamma(\alpha)^2 B(\alpha(k+1), 2(1-\alpha))}{\Gamma(2(1-\alpha))\,\Gamma(2\alpha-1)}\left[(a*b)_k - \alpha(k+1)\sum_{\ell=1}^{k} B(\alpha(\ell+2)-1,\, \alpha(k-\ell-1)+2)\,(b^{*2})_\ell\, c_{k-\ell}\right].$$

A Numerical illustration: (Fake) Stationary regime, α-fractional kernels $K_\alpha(t) = \frac{u^{\alpha-1}}{\Gamma(\alpha)}\mathbf{1}_{\mathbb{R}}(t)$:

Proposition (Existence and Properties of the function $\varsigma^2_{\alpha,\lambda,c}$ for $\alpha \in (\frac{1}{2}, \frac{3}{2})$)

The convergence radius of the fractional power series (40) that defines $\zeta_{\alpha,\lambda,c}$ is infinite and $\zeta_{\alpha,\lambda,c}$ is positive on $(0,+\infty]$ so that $\zeta_{\alpha,\lambda,c}$ is well-defined: The stabilizer $\varsigma^2_{\alpha,\lambda,c}$ exists as an non-negative function, such that:

1. $\lim_{t\to 0}\zeta_{\alpha,\lambda,c} = \begin{cases} 0 \text{ if } \alpha \le 1, \\ +\infty \text{ if } \alpha > 1, \end{cases}$ *and* $\lim_{t\to+\infty}\zeta_{\alpha,\lambda,c}(t) = \frac{\sqrt{c}\lambda}{\|f_{\alpha,\lambda}\|_{L^2(Leb_1)}}$.

We introduce an Euler-Maruyama scheme (41) on the time grid $t_k = t_k^n = \frac{kT}{n}$, $k = 0,\dots,n$, for the semi-integrated form (39), which write recursively:

$$\overline{X}_{t_k} = \frac{\theta_0}{\lambda} + (X_0 - \frac{\theta_0}{\lambda})R_\lambda(t_k) + \sum_{\ell=1}^{k}\frac{1}{\lambda}\int_{t_{\ell-1}}^{t_\ell} f_\lambda(t_k - s)\,\varsigma(t_\ell)\sigma(\overline{X}_{t_{\ell-1}})dW_s = g(t_k) + \frac{1}{\lambda}\sum_{\ell=1}^{k}\varsigma(t_\ell)\sigma(\overline{X}_{t_{\ell-1}})I_k^{n,l} \tag{41}$$

where the integrals $\left(I_k^{n,l} = \int_{t_{\ell-1}}^{t_\ell} f_\lambda(t_k - s)dW_s\right)_k$ can be simulated on the discrete grid $(t_k^n)_{0\le k\le n}$ by generating an independent sequence of gaussian vectors $G^{n,l}$, $l = 1\cdots n$ using the Cholesky decomposition of a well-defined covariance matrix C.

A Numerical illustration: Confluence of Stabilized Fractional-CIR Process. $\alpha \in (\frac{1}{2}, \frac{3}{2}) \subset (0, 2)$

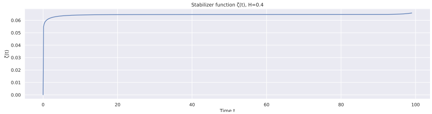

Figure: Graph of the stabilizer $t \to \varsigma_{\alpha,\lambda,c}(t)$ over time interval [0, T], T $=$ 100 for a value of the Hurst esponent $H = 0.4$, $\lambda = 0.2$, c $=$ 0.3.

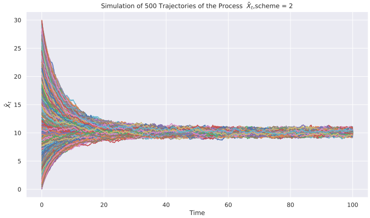

Figure: Confluence or Contraction from a [0,30]-Uniform Distribution.

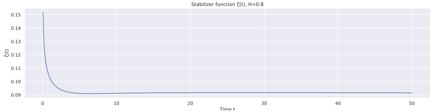

Figure: Graph of the stabilizer $t \to \varsigma_{\alpha,\lambda,c}(t)$ over time interval [0, T], T $=$ 50 for a value of the Hurst esponent $H = 0.8$, $\lambda = 0.2$, c $=$ 0.36.

A Numerical illustration: Fractional-CIR Process in the (Fake) Stationary regime with $\alpha \in (\frac{1}{2}, \frac{3}{2})$

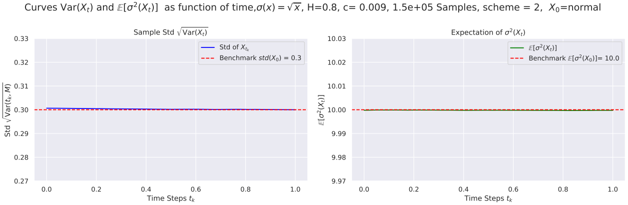

Figure: Graph of $t_k \mapsto \text{StdDev}(t_k, M)$ and $t_k \mapsto \mathbb{E}[\sigma^2(X_{t_k}, M)]$ over the time interval $[0, T]$, $T = 1$, $H = 0.8$, $\theta_0 = 2$, $\lambda = 0.2$, $v_0 = 0.09$, and $\text{StdDev}(X_0) = 0.3$, $\nu = 1$. Number of steps: $n = 800$, Simulation size: $M = 150000$.

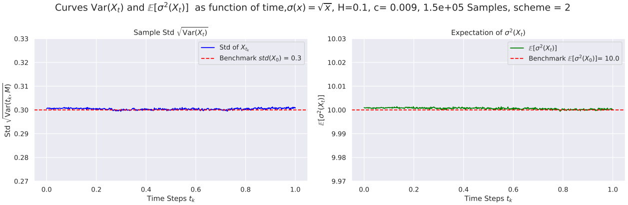

Figure: Graph of $t_k \mapsto \text{StdDev}(t_k, M)$ and $t_k \mapsto \mathbb{E}[\sigma^2(X_{t_k}, M)]$ over the time interval $[0, T]$, $T = 1$, $H = 0.1$, $\theta_0 = 2$, $\lambda = 0.2$, $v_0 = 0.09$, and $\text{StdDev}(X_0) = 0.3$, $\nu = 1$. Number of steps: $n = 800$, Simulation size: $M = 100000$.

The Stabilized Volterra Heston model and its characteristic functions.

In this setting, we define the process $X = (\log S, V)$, where S denotes the asset price and V its variance process, governed by

$$\frac{dS_t}{S_t} = \sqrt{V_t}\left(\sqrt{1-\rho^2}dW_t^{(1)} + \rho dW_t^{(2)}\right), \qquad S_0 \in (0,\infty), \tag{42}$$

and

$$V_t = V_0\phi(t) + \int_0^t K_\alpha(t-s)\left((\theta(s) - \lambda V_s)ds + \nu\varsigma(s)\sqrt{V_s}\, dW_s^{(2)}\right), \quad \varsigma = \varsigma_{\lambda,c}, \tag{43}$$

With

$$\forall\, t \geq 0, \quad \phi(t) = 1 - \lambda \int_0^t K_\alpha(t-s)\left(\frac{\theta(s)}{\mu_\infty} - 1\right) ds, \;\; c\lambda^2\left(1 - (\phi(t) - (f_{\alpha,\lambda} * \phi)_t)^2\right) = (f^2_{\alpha,\lambda} * \varsigma^2_{\alpha,\lambda,c})(t). \tag{44}$$

where the kernel K_α lies in $L^2_{\rm loc}(\mathbb{R}_+,\mathbb{R})$, $W = (W_1, W_2)$ is a two-dimensional standard Brownian motion with correlation $\rho \in [-1,1]$, and the θ a deterministic function, $\lambda, \nu \in \mathbb{R}_+$ such that V is at least a weak solution to the Volterra equation (3). More precisely, (43) can be rewritten:

$$V_t = V_0 - \left(V_0 - \frac{\mu_\infty}{\lambda}\right)\int_0^t f_{\alpha,\lambda}(t-s)\frac{\theta(s)}{\mu_\infty}ds + \frac{1}{\lambda}\int_0^t f_{\alpha,\lambda}(t-s)\varsigma_{\alpha,\lambda,c}(s)\sqrt{V_s}dW_s, \quad \lambda\,, \varsigma_{\alpha,\lambda,c}(t), \geq 0.$$

Once V is determined, the asset price process S follows accordingly. Moreover by applying Itô's formula, one can verify that for every $t \in [0, T]$, the log-price process satisfies

$$\log(S_t) = \log(S_0) + \int_0^t \sqrt{V_s}\left(\sqrt{1-\rho^2}dW_s^{(1)} + \rho dW_s^{(2)}\right) - \int_0^t \frac{V_s}{2}ds. \tag{45}$$

Hence, the process $X = (\log S, V)$ constitutes an affine Volterra process which evolves according to the system

$$\begin{aligned}\begin{pmatrix}\log(S_t)\\ V_t\end{pmatrix} &= \begin{pmatrix}\log(S_0)\\ V_0\phi(t)\end{pmatrix} + \int_0^t \begin{pmatrix}1 & 0\\ 0 & K_\alpha(t-u)\end{pmatrix}\left[\begin{pmatrix}0\\ \theta(u)\end{pmatrix} + \begin{pmatrix}0\\ 0\end{pmatrix}\log(S_u) + \begin{pmatrix}-\frac{1}{2}\\ -\lambda\end{pmatrix} V_u\right] du \\ &+ \int_0^t \begin{pmatrix}1 & 0\\ 0 & K_\alpha(t-u)\end{pmatrix}\begin{pmatrix}\sqrt{1-\rho^2} & \rho\\ 0 & \nu\varsigma(u)\end{pmatrix}\sqrt{V_u}\, dW_u, \quad t\in[0,T], \quad \varsigma = \varsigma_{\lambda,c}.\end{aligned} \tag{46}$$

We thus obtain that for the stabilized Volterra-Heston model the Riccati-Volterra equation (20) where we take $\mu(ds) = u\,\delta_0(ds) + f(s)\lambda_1(ds)$[1], in dimension 2, for any $u \in (\mathbb{C}^2_-)^*$ and $f \in L^1\left([0,T],(\mathbb{C}^2_-)^*\right)$ takes the form:

$$\begin{aligned}\psi_1(t) &= u_1 + \int_0^t f_1(s)ds,\\ \psi_2(t) &= u_2 K_\alpha(t) + \int_0^t K_\alpha(t-s)\Big(f_2(s) + \frac{1}{2}(\psi_1^2(s) - \psi_1(s))\\ &\quad + (\rho\nu\varsigma(T-s)\psi_1(s) - \lambda)\,\psi_2(s) + \frac{\nu^2}{2}\varsigma^2(T-s)\psi_2^2(s)\Big)ds, \quad t\in[0,T], \quad \varsigma=\varsigma_{\lambda,c}.\end{aligned} \tag{47}$$

[1] λ_1 denotes the Lebesgue measure on $(\mathbb{R}, \mathcal{B}or(\mathbb{R}))$

The Stabilized Volterra Heston model and its characteristic functions.

Proposition

Suppose that K_α satisfies condition (2)(i). Consider a stabilized Volterra Equation with $\lambda > 0$, $\mu_\infty \in \mathbb{R}$, where $\varsigma = \varsigma_{\lambda,c}$, assumed to be the unique continuous solution to Equation (33) for some $c > 0$ (so that condition $(E_{\lambda,c})$ is satisfied).

1. *The stochastic Volterra system (42)–(43) admits a $[0,+\infty)-$valued continuous weak solution $(\log S, V)$ with values in $\mathbb{R}\times\mathbb{R}_+$, for any initial state $(\log S_0, V_0) \in \mathbb{R}\times\mathbb{R}_+$. Moreover, the sample paths of V are $\left(\delta\wedge\vartheta\wedge\widehat{\theta}-\frac{1}{p}-\eta\right)$-Hölder pathwise continuous (up to P-indistinguability) for sufficiently small $\eta > 0$.*
2. *Let $u \in (\mathbb{C}^2)^*$ and $f \in L^1_{\rm loc}(\mathbb{R}_+, (\mathbb{C}^2)^*)$ such that*

   $$\mathrm{Re}\,\psi_1 \in [0,1],\ \mathrm{Re}\,u_2 \le 0,\ \text{and } \mathrm{Re}\,f_2 \le 0.$$

   where ψ_1 solves the first relation in (47). Then the second equation of the Riccati–Volterra equation (47) admits a unique global solution $\psi_2 \in L^2_{\rm loc}(\mathbb{R}_+, \mathbb{C}^*)$ with $\mathrm{Re}\,\psi_2 \le 0$. Furthermore, the exponential-affine representation (24) is valid with the measure $\mu(ds) = u\,\delta_0(ds) + f(s)\lambda_1(ds)$, in dimension 2 i.e. $u \in (\mathbb{C}^2_-)^*$ and $f \in L^1\left([0,T],(\mathbb{C}^2_-)^*\right)$.
3. *The process S solution of equation 42 is a true martingale and can be written:*

$$S_t = S_0 \exp\left(-\int_0^t \frac{V_s}{2}ds + \int_0^t \sqrt{V_s}\left(\sqrt{1-\rho^2}dW_s^{(1)} + \rho dW_s^{(2)}\right)\right), \quad t\in[0,T]. \tag{48}$$

The characteristic function of the Stabilized Rough Heston Model.

Corollary (Stabilized Rough Heston model (Here set $\mathbb{E}(V_0) := \bar{V}_0$).)

Let $\alpha \in (\frac{1}{2}, 1)$, and consider the $\alpha-$ fractional integration kernel $K_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t \in (0, T]$. Suppose $u \in (\mathbb{C}^2)^*$ and $f \in L^1\left([0, T], (\mathbb{C}^2)^*\right)$ satisfy the conditions $\Re\psi_1 \in [0, 1]$, $\Re u_2 \le 0$, and $\Re f_2 \le 0$, where $\psi_1 = u_1 + \int_0^\cdot f_1(s)\,ds$. Then there exists a unique function $\psi_2 \in L^2([0, T], \mathbb{C})$ solving the fractional Riccati equation

$$
\begin{aligned}
(D^\alpha \psi_2)(t) &= f_2(t) + \frac{1}{2}\left(u_1^2 - u_1 + 2u_1 \int_0^t f_1(s)\,ds + \left(\int_0^t f_1(s)\,ds\right)^2\right) \\
&\quad + \left(\rho\nu\varsigma(T-t)\left(u_1 + \int_0^t f_1(s)\,ds\right) - \lambda\right)\psi_2(t) + \frac{\nu^2}{2}\varsigma^2(T-t)\,\psi_2^2(t), \\
&\quad t \in [0, T], \quad \varsigma = \varsigma_{\lambda,c}, \qquad (I^{1-\alpha}\psi_2)(0) = u_2.
\end{aligned}
\tag{49}
$$

leading to the full Fourier–Laplace representation for the integrated log-price and variance:

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left(u_1 \log(S_T) + u_2 V_T + \int_0^T f_1(T-u)\log(S_u)\,du + \int_0^T f_2(T-u)V_u\,du\right)\right] \\
&= \exp\left(\varphi(T) + \left(u_1 + \int_0^T f_1(s)\,ds\right)\log(S_0) + \left(u_2\,\phi(T) + \int_0^T f_2(T-s)\,\phi(s)\,ds + \chi(T)\right)\bar{V}_0\right).
\end{aligned}
\tag{50}
$$

with $\chi(t) = \int_0^t \phi(t-s)\left(\frac{1}{2}(\psi_1^2(s) - \psi_1(s)) + (\rho\nu\varsigma(t-s)\psi_1(s) - \lambda)\,\psi_2(s) + \frac{\nu^2}{2}\varsigma^2(t-s)\psi_2^2(s)\right)ds$, $\quad t \in [0, T]$, and φ defined as $\varphi(t) = \int_0^t \theta(s)\psi_2(t-s)\,ds$, $\varphi(0) = 0$.

The characteristic function of the Stabilized Rough Heston Model.

Corollary (Stabilized Rough Heston model (Here set $\mathbb{E}(V_0) := \bar{V}_0$).)

In the particular case where $\phi \equiv 1$ (so that $\theta(t) = \theta_0 = \mu_\infty \ \forall t \geq 0$)[a], the above Fourier–Laplace representation simplifies to:

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left(u_1 \log(S_T) + u_2 V_T + \int_0^T f_1(T-u)\log(S_u)\,du + \int_0^T f_2(T-u)V_u\,du\right)\right] \\
&= \exp\left(\varphi(T) + \left(u_1 + \int_0^T f_1(s)\,ds\right)\log(S_0) + (I^{1-\alpha}\psi_2)(T)\bar{V}_0\right).
\end{aligned} \tag{51}
$$

with $D^\alpha = \frac{d}{dt} I^{1-\alpha}$ where D^α and $I^{1-\alpha}$ denote, respectively, the Riemann–Liouville fractional derivative of order α, and the Riemann–Liouville fractional integral of order $1-\alpha$.

[a] In which case $\theta(t) = \theta_0 = \mu_\infty$ for all $t \geq 0$, as a necessary and sufficient condition for the process to have a constant mean

Practitioner corner: The characteristic function of the stabilized rough Heston model can be computed by solving (inhomogeneous) Riccati equations, using an adaptation of the fast hybrid schemes proposed by Callegaro, Grasselli et Pagès. Plugging their numerical solution into (51) yields the characteristic function, from which standard Fourier methods allow the pricing of call and put options (Carr Madan, Lewis, etc.).

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Thanks For Your Attention!

Questions ?