On Inhomogeneous Fractional Square-Root Processes: Microstructural Foundation and Weak Stationarity Theory

Emmanuel Gnabeyeu[1] and Gilles Pagès[2] and Mathieu Rosenbaum[3]

[1,2]LPSM, SU&UPC Paris, France; [3]CMAP, École Polytechnique, Paris, France.

Talk at the *"Seminaire doctoral LPSM"*, Paris, France.

Wednesday 7th January, 2026

Groupe de Travail: Mathematique financière, Probabilités Numeriques - Sorbonne Université

The *short/rough* and *long-term memory* of Volterra Equations

From Volterra Heston model to the Stabilized Volterra Heston model.

In the Volterra Heston model, we define the process $X = (\log S, V)$, where S denotes the asset price and V its variance process, governed by

$$\frac{dS_t}{S_t} = \sqrt{V_t}\,\left(\sqrt{1-\rho^2}\,dW_t^{(1)} + \rho\,dW_t^{(2)}\right), \qquad S_0 \in (0,\infty), \tag{1}$$

$$V_t = V_0\phi(t) + \int_0^t K_\alpha(t-s)\Big((\theta(s) - \lambda V_s)\,ds + \nu\,\sqrt{V_s}\,dW_s^{(2)}\Big)\,. \tag{2}$$

where the kernel K_α lies in $L^2_{\rm loc}(\mathbb{R}_+,\mathbb{R})$, $W = (W_1, W_2)$ is a two-dimensional standard Brownian motion with correlation $\rho \in [-1,1]$, and the θ a deterministic function, $\lambda, \nu \in \mathbb{R}_+$ such that V is at least a weak solution to the equation (2).

Drawbacks:

- Exhibits distinct regimes for short and long maturities
- Due to the intrinsic **non-stationarity** of Volterra equations

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Solution:

- Introduce the **"fake stationary Volterra" Heston model**
- Ensures a consistent and time-homogeneous modeling framework

$$V_t = V_0\phi(t) + \int_0^t K_\alpha(t-s)\Big((\theta(s) - \lambda V_s)\,ds + \nu\,\varsigma(s)\sqrt{V_s}\,dW_s^{(2)}\Big)\,, \qquad \varsigma = \varsigma_{\lambda,c}\cdot, \tag{3}$$

$$\forall\, t \geq 0, \quad \phi(t) = 1 - \lambda\int_0^t K_\alpha(t-s)\left(\frac{\theta(s)}{\mu_\infty} - 1\right)ds, \quad c\lambda^2\Big(1 - \big(\phi(t) - (f_{\alpha,\lambda} * \phi)_t\big)^2\Big) = (f^2_{\alpha,\lambda} * \varsigma^2_{\alpha,\lambda,c})(t). \tag{4}$$

Affine Stochastic Volterra Integral Equation (Convolutive kernel)

Motivation / Key Questions:

- Raises important questions about the **well-posedness** (at least in the sense of weak solutions) and **positivity** of Volterra equations (3)
- Addressed in this paper via a **continuous-time scaling limit** of suitably time-modulated, rescaled, non-Markovian **self-exciting linear Hawkes processes**

Alternatives:

- Approximations+Compactness+ Skorokhod representation.
- Euler-Maruyama Approximation + Tightness.
- Martingale Problem Formulation.

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Our Approach provides, Positivity and uniqueness in law:

In this work, we are chiefly interested in the the **well-posedness and positivity** of stochastic convolution Volterra integral equation of the form

$$X_t = X_0\phi(t) + \int_0^t K(t-s)(\theta(s) - \lambda X_s)ds + \nu \int_0^t K(t-s)\varsigma(s)\sqrt{X_s}\, dW_s, \quad X_0 \perp\!\!\!\perp W, \tag{5}$$

Entitled **Inhomogeneous Fractional Cox-Ingersoll-Ross Process**

On Wellposedness of Generalized Fractional Volterra square root/CIR SDE

Remark (From Hawkes Processes to Inhomogeneous Fractional Cox-Ingersoll-Ross Process)

From

$$\Lambda_t^T = \mu^T(t) + \sum_{\tau_i < t} \varphi^T(t - \tau_i) = \mu^T(t) + \int_0^t \varphi^T(t - s) dN_s^T \tag{6}$$

to

$$\Lambda_t^* = X_0 \phi(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} \lambda(\theta(s) - \Lambda_s^*)\, ds + \frac{\lambda \nu}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} \varsigma(s) \sqrt{\Lambda_s^*}\, dW_s.$$

via a Rescaling in time and space of the intensity process (Λ_t^T) *and asymptotic limit as* $T \to +\infty$

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Ideas.

1. Time-inhomogeneous volatility in the limiting diffusion induced via a time-dependent rescaling factor $\omega_T(t) = \frac{1-a_T}{\tilde{\mu}^T(t/T)}$, allowing control through a bounded function ς. Define $\Lambda_t^{*T} := \omega_T(t) \Lambda_{tT}^T$.

□

We rather consider a standard Hawkes process (N, Λ) defined on the whole real line with intensity having an initial condition:

$$\Lambda_t = Z_0(t) + \mu(t) + \int_{(0,t)} \varphi(t-s) N(ds), \tag{7}$$

The function $Z_0 \in L^1_{\text{loc}}(\mathbb{R}^+; \mathbb{R}^+)$ is a random function evolving deterministically for $t \geq 0$

The Hawkes process N has the predictable quadratic variation $\mathcal{I}_t^\Lambda = \int_0^t \Lambda_s \, ds$, so that the compensated point process $M := N - N_0 - \mathcal{I}^\Lambda$ is a $(\mathcal{F}_t)$-martingale by the very definition in equation (7) of the intensity Λ.

For every $\lambda \in \mathbb{R}$, the λ- *resolvent* R_λ associated with the kernel φ is defined as the unique solution (if it exists) to:

$$\forall\, t \geq 0, \quad R_\lambda(t) + \lambda \int_0^t \varphi(t-s) R_\lambda(s) ds = 1 \implies R_\lambda = \sum_{k \geq 0} (-1)^k \lambda^k (\mathbf{1} * \varphi^{k*}). \tag{8}$$

Theorem (Martingale representation)

The intensity Λ admits the representation $\Lambda_t = Z_0(t) + \mu(t) + (R'_{-1} * (Z_0 + \mu))_t + (R'_{-1} \overset{M}{*} \mathbf{1})_t$, $\quad t \geq 0$ or, equivalently,

$$\Lambda_t = Z_0(t) + \mu(t) - \int_0^t f_{-1}(t-s)(Z_0(s) + \mu(s))ds - \int_0^t f_{-1}(t-s) dM_s, \quad t \geq 0, \quad \textit{where} \quad f_\lambda = -R'_\lambda \quad \forall \lambda \in \mathbb{R}$$

Moreover if $R'_{-1}, \mathcal{I}^\varphi \in \mathcal{L}^2_{loc}(\mathbb{R}_+, Leb_1)$, the expected intensity (assume to be bounded) and the expected number of events are given by, respectively,

$$\mathbb{E}[\Lambda_t] = \mathbb{E}[Z_0(t)] + \mu(t) + (R'_{-1} * (\mathbb{E}[Z_0(\cdot)] + \mu(\cdot)))_t, \quad \mathbb{E}[N_t] = \mathbb{E}[\mathcal{I}_t^\Lambda] = \int_0^t (\mathbb{E}[Z_0(s)] + \mu(s))ds + ((\mathbb{E}[Z_0(\cdot)] + \mu(\cdot)) * \mathcal{I}^{R'_{-1}})_t \tag{9}$$

where

$$\mathcal{I}_t^{R'_{-1}} := \int_0^t R'_{-1}(s) ds = R_{-1}(t) - R_{-1}(0).$$

From Hawkes processes to Inhomogeneous Volterra CIR Equation:

Asymptotic Hawkes Setting

- Let $(N_t^T)_{t\geq 0}$ be a sequence of one-dimensional Hawkes processes indexed by $T > 0$, observed over $[0, T]$, and defined on the filtered probability space $(\Omega^T, \mathcal{F}^T, \mathbb{P}^T, (\mathcal{F}_t^T)_{t\in[0,T]})$ with $T \to \infty$.
- The intensity process $(\Lambda_t^T)_{t\geq 0}$ is given by:

$$\Lambda_t^T = Z_0^T(t) + \mu^T(t) + \int_0^t \varphi^T(t-s)\, dN_s^T$$

  where:
  - $\mu^T(t)$ is the non-negative baseline intensity.
  - φ^T is a memory kernel with $\|\varphi^T\|_1 < \infty$,
  - $Z_0^T(t) = \mathbb{E}[\Lambda_0^T]\Phi^T(t)$ with $\Phi^T(t) = \int_t^\infty \varphi^T(s)\, ds.$ is the initial condition.
- Introducing a positive real-valued random variable Λ_0^{*T}, the intensity can be rewritten as:

$$\Lambda_t^T = \Lambda_0^{*T}\Phi^T(t) + \mu^T(t) + \int_0^t \varphi^T(t-s)\, dN_s^T$$

Main Assumption:

Assumption (Standing Assumptions)

1. *We are working in the nearly unstable heavy tail case since* $\Phi(0) = 1$. *We assume for* $t \in \mathbb{R}^+$, $\varphi^T(t) = a_T \varphi(t)$, *where* $(a_T)_{T \geq 0}$ *is a sequence of positive numbers converging to* 1 *such that for all* T, $a_T < 1$ *and* φ *is a non-negative measurable and completely monotonic function such that* $\Phi(0) = \|\varphi\|_{L_1} = 1$.
2. *Furthermore, the function* φ *has a regular varying tail, i.e. there exists some* $\alpha \in (0,1)$ *and some positive constant* $C > 0$, *such that:* $\lim_{x \to +\infty} \alpha x^{\alpha} \Phi(x) = C$
3. *Set* $\delta = \frac{C\Gamma(1-\alpha)}{\alpha}$, *and for* $T > 0$, *consider* $\theta_0 \in \mathbb{R}_+$, *then a deterministic function* $\tilde{\mu}^T : [0, T] \to \mathbb{R}_+$, *uniformly continuous on the interval* $[0,1]$ *such that* $\tilde{\mu}^T(0) = \frac{\mu^T(0)}{\theta_0} = \frac{\mu^T}{\theta_0}$ *so that* $\forall t > 0, \lim_{T \to +\infty} \frac{\mu^T}{\tilde{\mu}^T(\frac{t}{T})} = \theta_0$ *and assume there exists some positive constants* λ, *a non-negative bounded borel function* ς *and a continuous function* $\phi \in \mathcal{L}^1_{\mathbb{R}_+}(Leb_1)$ *such that uniformly as* $T \to \infty$*:*

$$T^{\alpha}(1 - a_T) \to \lambda\delta; \quad \forall t > 0,\ T^{1-\alpha} \frac{(\mu^T)^2}{\theta_0^2 \tilde{\mu}^T(\frac{t}{T})} \to \frac{\lambda}{\delta \varsigma^2(t)} \quad \text{and} \quad T^{\alpha} \frac{a_T(1 - a_T)}{\tilde{\mu}^T(\frac{t}{T})} \to \phi(t).$$

4. *There exist a random variable* $\Lambda_0^* \in L^1(\mathbb{R}_+)$, *weak limit of the sequence* $(T^{1-\alpha} {\Lambda_0^*}^T)_{T \geq 0}$ *as* $T \to +\infty$ *i.e.*

$$\lim_{T \to +\infty} T^{1-\alpha} {\Lambda_0^*}^T = \Lambda_0^*$$

Intuitions and Main Heuristic Convergence Results

Let's write $\Lambda_t^{*T} := \omega_T(t)\Lambda_{tT}^T = \frac{1-a_T}{\tilde{\mu}^T(\frac{t}{T})}\Lambda_{tT}^T$

$$\Lambda_t^{*T} = \Lambda_0^{*T}\left(\Phi^T(tT) + (R'^T_{-1} * \Phi^T)(tT)\right)\frac{1-a_T}{\tilde{\mu}^T(\frac{t}{T})} + \frac{\mu^T}{\tilde{\mu}^T(\frac{t}{T})}\int_0^t T(1-a_T)R'^T_{-1}(T(t-s))\,\frac{\mu^T(Ts)}{\mu^T}\,ds$$

$$+ \underbrace{(1-a_T)\frac{\mu^T(tT)}{\tilde{\mu}^T(\frac{t}{T})}}_{\underset{T\to+\infty}{\to} 0} + \underbrace{\frac{\mu^T}{\tilde{\mu}^T(\frac{t}{T})}\int_0^t \frac{(1-a_T)}{\mu^T}R'^T_{-1}(T(t-s))\,dM^T_{Ts}}_{:=K_t^T}, \quad \forall t \in [0, t_0]$$

$$K_t^T = \frac{\mu^T}{\tilde{\mu}^T(\frac{t}{T})}\int_0^t \underbrace{\sqrt{\frac{\tilde{\mu}^T(\frac{s}{T})}{T(1-a_T)(\mu^T)^2}}}_{\to\frac{\varsigma(s)}{\lambda}}\,\underbrace{T(1-a_T)R'^T_{-1}(T(t-s))}_{m^T(ds)\to m^*(ds)}\,\sqrt{\Lambda_s^{*T}}\,dW_s^T$$

where we set, $W_t^T := \frac{1}{\sqrt{T}}\int_0^{tT}\frac{dM_s^T}{\sqrt{\Lambda_s^T}}$, $\quad\forall t \in [0,1]$ is cvgce to a BM by Levy's characterization of Brownian motion.

Assumption

The baseline intensity $\mu^T(t)$ *is given by* $\mu^T(t) = \mu^T\zeta^T(t)$ *with* $\zeta^T(t) = 1 - \int_0^t \varphi^T(t-u)\left(1 - \frac{\theta(\frac{u}{T})}{\theta_0}\right)du$ *for a deterministic function* θ*, continuous on* $\mathbb{R}_+^*$*.*

Intuitions and Main Heuristic Convergence Results

Proposition (The limits for the integrated rescaled resolvent function)

The finite measure on $\mathbb{R}_+$ with density $m^T(ds) := T(1-a_T)R'^T_{-1}(Ts)\,ds$ converges weakly to the finite measure on $\mathbb{R}_+$ with density $m^*(ds) := f_{\alpha,\lambda}(s)\,ds$ so that at least heuristically, under Assumption 3:

1. *Convergce of rescaled integrated resolvent* : $\sup_{t\geq 0}\left| \int_0^t T(1-a_T)R'^T_{-1}(Tu)\,du - \int_0^t f_{\alpha,\lambda}(u)\,du\right| \underset{T\to+\infty}{\to} 0$
2. $\sup_{t\geq 0}\left| T^{\alpha}\frac{(1-a_T)}{\tilde{\mu}^T(\frac{t}{T})}\Big(\Phi^T(tT) + (R'^T_{-1} * \Phi^T)(tT)\Big) - \left(\phi(t) - \int_0^t f_{\alpha,\lambda}(t-s)\phi(s)ds\right)\right| \underset{T\to+\infty}{\to} 0$
3. $\sup_{t\geq 0}\left| \int_0^t T(1-a_T)R'^T_{-1}(T(t-s))\theta(s)\,ds - \int_0^t f_{\alpha,\lambda}(t-s)\theta(s)ds\right| \underset{T\to+\infty}{\to} 0.$

Where $f_{\alpha,\lambda}$ is the Mittag-Leffler density function and $R_{\alpha,\lambda}(t)$ denotes the corresponding resolvent.

The function $f_{\alpha,\lambda}$ is defined on $(0,+\infty)$ by

$$f_{\alpha,\lambda}(t) := -R'_{\alpha,\lambda}(t) = \alpha\lambda t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha}) = \lambda t^{\alpha-1}\sum_{k=0}^{\infty}\frac{(-1)^k\lambda^k t^{\alpha k}}{\Gamma(\alpha(k+1))}, \tag{1.14}$$

is a probability density, called the *Mittag–Leffler density* and $E_{\alpha,\alpha}$ is the so called *Mittag–Leffler function*.

Heuristically, for the stochastic integral term

$$K^T_t \underset{T\to+\infty}{\to} \frac{1}{\lambda}\int_0^t f_{\alpha,\lambda}(t-s)\varsigma(s)\sqrt{\Lambda^*_s}\,dW_s$$

Heuristic Convergence Results for Heavy-Tailed Nearly Unstable Hawkes Processes

Therefore, taking the limit as $T \to \infty$ and owing to our Assumption 3 and 4 we expect the limiting process Λ_t^* to be the solution of the following stochastic volterra integro-differential equation:

$$\Lambda_t^* = \Lambda_0^* \phi(t)\big(\phi(t) - \int_0^t f_\lambda(t-s)\phi(s)ds\big) + \int_0^t f_{\alpha,\lambda}(t-s)\theta(s)ds + \frac{1}{\lambda}\int_0^t f_{\alpha,\lambda}(t-s)\varsigma(s)\sqrt{\Lambda_s^*}dW_s. \tag{10}$$

Theorem

Any solution to the stochastic Volterra equation (10) *can be equivalently represented as*

$$\Lambda_t^* = \Lambda_0^* \phi(t) + \int_0^t K_\alpha(t-s)\lambda(\theta(s) - \Lambda_s^*)ds + \int_0^t K_\alpha(t-s)\varsigma(s)\sqrt{\Lambda_s^*}dW_s, \quad \Lambda_0^* \perp\!\!\!\perp W, \quad t \geq 0. \tag{11}$$

where $K_\alpha := u \to \frac{u^{\alpha-1}}{\Gamma(\alpha)}$ *is the fractional integration kernel.*

Proof.

This is a direct consequence of (Gnabeyeu, 2025a, Proposition 3.0.1 and Remark 3.0.2) ☐

Formal Proof by Functional Limits Theorem: "À la Jacod–Shiryaev"

We introduce the family of rescaled processes: $X^T := \left\{X_t^T = \left(\tilde{\mathcal{I}}_t^{\Lambda^* T}, \tilde{N}_t^{*T}, \tilde{M}_t^{*T}, \tilde{M}_t\right) : t \geq 0\right\}$ where $\forall t > 0$:

$$\tilde{N}_t^{*T} = \frac{(1-a_T)}{T}\int_0^{tT} \frac{1}{\tilde{\mu}^T(\frac{s}{T^2})} dN_s^T, \quad \tilde{\mathcal{I}}_t^{\Lambda^* T} = \frac{(1-a_T)}{T}\int_0^{tT} \frac{1}{\tilde{\mu}^T(\frac{s}{T^2})} \Lambda_s^T ds, \quad \tilde{M}_t^{*T} = \sqrt{\frac{1-a_T}{T}}\int_0^{tT} \frac{dM_s^T}{\sqrt{\tilde{\mu}^T(\frac{s}{T^2})}}$$

and $\tilde{M}_t^T := \int_0^t \varsigma^T(s)\, d\tilde{M}_s^{*T}$ where we set $\varsigma^T(t) := \frac{\lambda}{\nu}\sqrt{\frac{1}{T(1-a_T)\tilde{\mu}^T(\frac{t}{T})}}$ so that $\tilde{M}_t^T = \frac{\lambda}{\nu T}\int_0^{tT} \frac{dM_s^T}{\tilde{\mu}^T(\frac{s}{T^2})}$.

Proposition (A priori estimates and $\mathbb{C}$-tightness of $X^T = \left(\tilde{\mathcal{I}}^{\Lambda^* T}, \tilde{N}^T, \tilde{M}^{*T}, \tilde{M}^T\right)$)

Let $t_0 > 0$. *Under Assumption 3, the sequence* $X^T = \left(\tilde{\mathcal{I}}^{\Lambda^* T}, \tilde{N}^T, \tilde{M}^{*T}, \tilde{M}^T\right)$ *satisfies the following:*

1. *The sequence* $(X^T)_{T>0}$ *is $\mathbb{C}$-tight in the Skorokhod J_1-topology on* $\mathcal{D}([0,t_0], \mathbb{R}_+^2 \times \mathbb{R}^2)$. *As a result, any limiting process lies in* $\mathcal{C}([0,t_0], \mathbb{R}_+^2 \times \mathbb{R}^2))$, *i.e., it has continuous paths.*
2. *The processes* $\tilde{\mathcal{I}}^{\Lambda^* T}$ *and* $\tilde{N}^T$ *become asymptotically indistinguishable in probability:*$\sup_{t\in[0,t_0]}\left|\tilde{\mathcal{I}}_t^{\Lambda^* T} - \tilde{N}_t^T\right| \xrightarrow[T\to\infty]{\mathbb{P}} 0$.
3. *Futhermore If* $X = \left(\tilde{\mathcal{I}}^{\Lambda^*}, \tilde{N}, \tilde{M}^*, \tilde{M}\right)$ *is any limit point of* $(X^T)_{T>0}$, *then* $\tilde{M}^*$ *is a continuous martingale with quadratic variation* $\langle\tilde{M}^*\rangle = \tilde{N} = \tilde{\mathcal{I}}^{\Lambda^*}$. $\tilde{M}$ *is a continuous martingale* $\tilde{M}_t := \int_0^t \varsigma(s)\, d\tilde{M}_s^*$, $\langle\tilde{M}\rangle = \int_0^\cdot \varsigma^2(s) d\langle\tilde{M}^*\rangle_s = \tilde{\mathcal{I}}^{\varsigma^2\Lambda^*}$.
4. *The sequence* $(\Lambda^{*T})_{T>0}$ *is uniformly integrable and $\mathcal{C}$-tight in the Skorokhod topology on* $\mathcal{D}([0,t_0], \mathbb{R}_+)$.

Dynamics of the Limit Points

From what follows, as $T \to \infty$,

$$\Lambda^{*T} \overset{\mathcal{L}-\mathcal{S}}{\Rightarrow} \Lambda^* \quad \text{in } D([0,t_0];\mathbb{R}_+) \quad \Rightarrow \quad \tilde{\mathcal{I}}^{\Lambda^{*T}} \overset{\mathcal{L}-\mathcal{S}}{\Rightarrow} \tilde{\mathcal{I}}^{\Lambda^*} := \int_0^{\cdot} \Lambda^*_s \, ds \quad \text{in } D([0,t_0];\mathbb{R}_+). \tag{12}$$

Theorem (Characterization of the Limiting Processes and Moment Control)

Let $t_0 > 0$. As $T \to +\infty$, under Assumption 3, $(X^T_t)_{t\in[0,t_0]} \overset{\mathcal{L}-\mathcal{S}}{\Rightarrow} (X_t)_{t\in[0,t_0]}$ i.e. the sequence $(X^T_t)_{t\in[0,t_0]}$ converges in law for the Skorohod topology on $[0,t_0]$ to $X := \left\{X_t = \left(\tilde{\mathcal{I}}^{\Lambda^*}_t, \tilde{N}_t, \tilde{M}^{*T}_t, \tilde{M}_t\right) : t \geq 0\right\}$ satisfying the following properties:

- *Almost surely cvgce i.e.* $\lim_{T\to\infty} \sup_{t\in[0,t_0]} \left(|\tilde{\mathcal{I}}^{\Lambda^{*T}}_t - \tilde{\mathcal{I}}^{\Lambda^*}_t| + |\tilde{N}^T_t - \tilde{N}^*_t| + |\tilde{M}^{*T}_t - \tilde{M}^*_t| + |\tilde{M}^T_t - \tilde{M}_t|\right) = 0 \quad a.s.$
- *There exists a brownian motion W such that: $\tilde{M}^*_t = \int_0^t \sqrt{\Lambda^*_s} dW_s$ and $\tilde{M}_t := \int_0^t \varsigma(s) \sqrt{\Lambda^*_s} dW_s$.*
- *$\tilde{\mathcal{I}}^{\Lambda^*}_t = \tilde{N}_t = \int_0^t \Lambda^*_s ds$ i.e. Λ^* is the derivative of $\tilde{N}$ and the unique continuous weak solution of the stochastic Volterra integral equation on $[0,t_0]$ where $K_\alpha := u \to \frac{u^{\alpha-1}}{\Gamma(\alpha)}$.*

$$\Lambda^*_t = \Lambda^*_0 \phi(t) + \int_0^t K_\alpha(t-s)\lambda(\theta(s) - \Lambda^*_s)ds + \nu \int_0^t K_\alpha(t-s)\varsigma(s)\sqrt{\Lambda^*_s} dW_s, \quad X_0 \perp\!\!\!\perp W,$$

  In particular, the sequence of rescaled Hawkes processes converges uniquely in law.

- *Furthermore, the process Λ^* is non-negative and has we have the following moment estimate $\forall\, p > 0$:*

$$\sup_{t\in[0,t_0]} \mathbb{E}\left[|\Lambda^*_t|^p\right] \leq C_{p,t_0} \cdot (1 + \|\phi\|_{t_0} \mathbb{E}[|\Lambda^*_0|^p]) \, .$$

Regularity and Maximal Inequality for the Limit Points

Assumption (Integrability and Uniform Hölder Continuity)

Let $\lambda, c > 0$. *Assume the kernel* K_α *is such that its* λ*-resolvent* $R_{\alpha,\lambda}$ *and its derivative* $-f_{\alpha,\lambda}$ *satisfy:*

(i) **Integrability:**

$$\int_0^{+\infty} f_{\alpha,\lambda}^{2\beta}(u)\,du < +\infty \quad \textit{for some } \beta \geq 1,$$

so that $f_\lambda \in \mathcal{L}^2(Leb_1)$.

(ii) **Hölder Continuity:** *There exists* $\vartheta \in (0,1]$, $C < +\infty$ *such that:* $\left(\vartheta \in \left(0, \alpha - \frac{\mathbf{1}_{i=2}}{i}\right)\right)$

$$\max_{i=1,2}\left[\int_0^{+\infty} |f_{\alpha,\lambda}(u+\bar{\delta}) - f_{\alpha,\lambda}(u)|^i\,du\right]^{\frac{1}{i}} \leq C\bar{\delta}^{\vartheta}.$$

(iii) **Moment and Regularity Bounds:** *For some* $\delta > 0$, *for any* $p > 0$ *and* $T > 0$,

$$\mathbb{E}\left(\sup_{t\in[0,T]} |\Lambda_0^*\phi(t)|^p\right) < +\infty, \quad \mathbb{E}\left|\Lambda_0^*\phi(t') - \Lambda_0^*\phi(t)\right|^p \leqslant C_{T,p}\left(1 + \mathbb{E}\left[\sup_{t\in[0,T]} |\Lambda_0^*\phi(t)|^p\right]\right)|t'-t|^{\delta p}.$$

Regularity and Maximal Inequality for the Limit Points

Theorem (Regularity and Maximal Inequality for Generalized Volterra Fractional Diffusion)

Let Λ^* *be any weak solution to the stochastic Volterra equation* (10). *Then the following hold:*

1. *The process* Λ^* *is non-negative and has Hölder regularity* $\delta \wedge \vartheta \wedge \frac{\beta-1}{2\beta} - \epsilon$ *for any* $\epsilon > 0$ *i.e. the process* Λ^* *is almost surely Hölder continuous of any order strictly less than* $\delta \wedge \vartheta \wedge \frac{\beta-1}{2\beta}$.
2. **a-Hölder seminorm:** *For any* $a \in \left(0, \delta \wedge \theta \wedge \frac{\beta-1}{2\beta}\right)$ *and* $p \geq 0$, *there exists positive real constant* $C_{a,p,t_0} = C_{a,\delta,\vartheta,\beta,p,t_0,f_{\alpha,\lambda}}$ *such that for any* $t_0 \geq 0$,

$$\left\| \sup_{s \neq t \in [0,t_0]} \frac{|\Lambda^*_t - \Lambda^*_s|}{|t-s|^a} \right\|_p^p = \mathbb{E}\left[\sup_{s \neq t \in [0,t_0]} \frac{|\Lambda^*_t - \Lambda^*_s|^p}{|t-s|^{ap}} \right] < C_{a,p,t_0}\left(1 + \|\phi\|_{t_0} \mathbb{E}[|\Lambda^*_0|^p]\right) \tag{13}$$

3. **Sup-norm:** *In particular for each* $p \geq 0$, *there exists a constant* $C > 0$ *such that for any* $T > 0$ [a],

$$\mathbb{E}\left[\sup_{t \in [0,t_0]} |\Lambda^*_t|^p \right] \leq C'_{a,p,T}\left(1 + \|\phi\|_{t_0} \mathbb{E}[|\Lambda^*_0|^p]\right), \left\| \sup_{t \in [0,t_0]} |\Lambda^*_t| \right\|_p \leq K'_{a,p,T}\left(1 + \|\phi\|_{t_0} \mathbb{E}[|\Lambda^*_0|^p]\right). \tag{14}$$

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[a] Kolmogorov's continuity criterion or Garsia-Rodemich-Rumsey inequality/Lemma

Inhomogeneous Fractional Cox–Ingersoll–Ross Process: Investigating Stationarity

$$X_t = X_0\phi(t) + \int_0^t K(t-s)(\theta(s) - \lambda X_s)\, ds + \nu \int_0^t K(t-s)\varsigma(s)\sqrt{X_s}\, dW_s, \quad X_0 \perp\!\!\!\perp W. \tag{15}$$

1. **As a consequence of Wiener–Hopf and stochastic/ordinary Fubini's theorems, equation (15) reads:**

$$X_t = X_0\big(\phi(t) - (f_\lambda * \phi)(t)\big) + \frac{1}{\lambda}\int_0^t f_\lambda(t-s)\mu(s)\, ds + \frac{\nu}{\lambda}\int_0^t f_\lambda(t-s)\sqrt{X_s}\, dW_s. \tag{16}$$

Looking for stationarity!

- Either in the classical sense, where the distribution is invariant under time shifts.
- True Volterra equations have no stationary regime!

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Looking for stationarity!

- Either in the classical sense, where the distribution is invariant under time shifts.
- True Volterra equations have no stationary regime!
- Towards stationarity in a weaker sense.

② **Standing assumption on the kernel** K**:**

Assumption (λ-resolvent R_λ)

$$(\mathcal{K}) \quad \left\{ \begin{array}{ll} (i) & f_\lambda \in \mathcal{L}^p_{loc}(\mathbb{R}_+, Leb_1),\ \textit{for } p \geq 1\ ,\ \textit{for}\ \ t > 0,\ L_{f_\lambda}(t) \neq 0\ dt - a.e.,\ \textit{where } f_\lambda := -R'_\lambda, \\ (ii) & \phi \in \mathcal{L}^1_{\mathbb{R}_+}(Leb_1),\ \textit{is a continuous function satisfying } \lim_{t\to\infty}\phi(t) = \phi_\infty, \\ (iii) & \theta \textit{ is a } C^1\textit{-function such that } \|\theta\|_{\sup} < \infty \textit{ and } \lim_{t\to+\infty}\theta(t) = \mu_\infty \in \mathbb{R}. \end{array} \right. \tag{17}$$

Fake Stationarity of Time-Inhomogeneous Affine Volterra Process: Finite-time.

Definition (Fake Stationary Regime of type I [2] G.Pagès 2024)

Let $(X_t)_{t\geq 0}$ be a solution to the scaled Volterra equation in its form (11) starting from any $X_0 \in L^2(\mathbb{P})$. Let $\sigma(t,x) = \varsigma(t)\sigma(x)$ in equation (11), where $\sigma(x) = \nu\sqrt{x}$.

1. The process $(X_t)_{t\geq 0}$ exhibit a fake stationary regime of type I if it has constant mean, variance, and $\mathbb{E}[\sigma^2(X_t)]$ i.e.:

$$\forall\, t \geq 0, \quad \mathbb{E}[X_t] = c^{\text{ste}} \quad \mathrm{Var}(X_t) = c^{\text{ste}} = v_0 \geq 0 \quad \text{and} \quad \bar{\sigma}^2(t) := \mathbb{E}[\sigma^2(X_t)] = c^{\text{ste}} := \bar{\sigma}_0^2 \geq 0. \tag{18}$$

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Theorem (Time-Dependent Volatility σ. Let $\sigma(t,x)=\varsigma(t)\sigma(x)$ in equation (11))

Assume that $X_0\in L^2(\mathbb{P})$ *with* $\mathbb{E}[X_0]=\frac{\mu_\infty}{\lambda}$. *Then, a necessary condition for the relations 18 to be satisfied is that* $\forall\, t\ge 0$, *:*

$$\phi(t)=1-\lambda\int_0^t K(t-s)\left(\frac{\theta(s)}{\mu_\infty}-1\right)ds,\ \textit{so that}\ X_t=X_0-\left(X_0-\frac{\mu_\infty}{\lambda}\right)\int_0^t f_\lambda(t-s)\frac{\theta(s)}{\mu_\infty}ds+\frac{1}{\lambda}\int_0^t f_\lambda(t-s)\varsigma(s)\sigma(X_s)dW_s. \tag{19}$$

and the triplet $(v_0,\bar\sigma_0^2,\varsigma(t))$*, where* $v_0=Var(X_0)$ *and* $\bar\sigma_0^2=\mathbb{E}[\sigma^2(X_0)]$*, must satisfy the following functional equation:*

$$(E_{\lambda,c}):\quad \forall\, t\ge 0,\quad c\lambda^2\big(1-(\phi(t)-(f_\lambda*\phi)_t)^2\big)=(f_\lambda^2*\varsigma^2)(t)\quad \textit{where}\quad c=\frac{v_0}{\bar\sigma^2}\quad \textit{and thus}\quad \varsigma=\varsigma_{\lambda,c}. \tag{20}$$

Long run behavior $\sigma(t,x) = \nu\varsigma_{\lambda,c}(t)\sqrt{x}$: functional weak asymptotics

Theorem (Long run theorem)

If $(X_t)_{t\geq 0}$ of the SVIE (**??**) has a fake stationary regime of type I, starting from a random variable $X_0 \in L^2(P)$ with mean $\frac{\mu_\infty}{\lambda}$ and variance v_0,

(a) The family of shifted processes $(X_{t+u})_{u\geq 0}$ is $\mathcal{C}$-tight and uniformly square integrable as $t \to +\infty$. There exists a process X^∞ with a $\left(\delta \wedge \vartheta \wedge \frac{\beta-1}{2\beta} - \frac{1}{p} - \eta\right)$-Hölder pathwise continuous sample paths for sufficiently small $\eta > 0$ such that

$$(X_{t+u})_{t\geq 0} \Rightarrow (X^\infty_t)_{t\geq 0} \quad \text{weakly in } C(\mathbb{R}_+;\mathbb{R}) \text{ as } u \to \infty.$$

(b) **Functional weak long-run behavior.** For $t_1, t_2 \geq 0$, $t_1 \leq t_2$,

$$\mathrm{Cov}(X_{t+t_1}, X_{t+t_2}) \overset{t\to+\infty}{\longrightarrow} \bar{C}_{f_\lambda}(t_1, t_2) := \frac{v_0}{\int_0^{+\infty} f_\lambda^2(s)ds} \int_0^{+\infty} f_\lambda(t_2 - t_1 + u) f_\lambda(u)du. \tag{21}$$

Thus, under any limiting distribution P, on the canonical space $\Omega_0 := \mathcal{C}(\mathbb{R}_+, \mathbb{R})$, the canonical process

$$Y_t(\omega) = \omega(t),\ \ \omega \in \Omega_0 \text{ is a } (weak)L^2\text{-stationary process with mean } \frac{\mu_\infty}{\lambda} \text{ and covariance function } \bar{C}_{f_\lambda}(s,t),\ \ s,t \geq 0.$$

Applications: Computing the Stabilizer $\varsigma_{\alpha,\lambda,c}$ for the α-fractional kernels $K_\alpha(t) = \frac{u^{\alpha-1}}{\Gamma(\alpha)}\mathbf{1}_{\mathbb{R}}(t)$:

For the numerical illustration, we consider α-fractional kernels with $\alpha \in \left(\frac{1}{2}, 1\right)$ ("rough models") or $\alpha \in \left(1, \frac{3}{2}\right)$ ("long memory models"), within the setting where $\phi(t) = \phi(0) = 1 \quad$ for all $t \geq 0 \quad$ a.s.
In this case, the equation simplifies in the so-called fake stationarity regime (i.e., $\theta(t) = \theta_0$ and $\sigma(x) = \nu\sqrt{x}$) as follows:

$$X_t = \frac{\theta_0}{\lambda} + \left(X_0 - \frac{\theta_0}{\lambda}\right)\left(1 - \int_0^t f_{\alpha,\lambda}(s)\, ds\right) + \frac{1}{\lambda}\int_0^t f_{\alpha,\lambda}(t-s)\varsigma_{\alpha,\lambda,c}(s)\sigma(X_s)dW_s. \tag{22}$$

$$(E_{\lambda,c}): \quad \forall\, t \geq 0, \quad c\lambda^2\left(1 - (1 - \int_0^t f_{\alpha,\lambda}(s)\, ds)^2\right) = (f^2_{\alpha,\lambda} * \varsigma^2_{\alpha,\lambda,c})(t) \quad \textit{where} \quad c > 0. \tag{23}$$

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Regular Variation (Tauberian theorem) on Laplace transforms of the functional equation above suggests to search $\varsigma^2_{\alpha,\lambda,c}(t)$ as an expansion of the form (*Power Series Ansatz*): *Set* $a_k = \frac{1}{\Gamma(\alpha k+1)}, b_k = \frac{1}{\Gamma(\alpha(k+1))}, \ k \geq 0.$

$$\varsigma^2_{\alpha,\lambda,c}(t) = c\lambda^{2-\frac{1}{\alpha}}\varsigma^2_{\alpha}(\lambda^{\frac{1}{\alpha}}t) \quad \textit{where} \quad \varsigma^2_{\alpha}(t) := 2\,t^{1-\alpha}\sum_{k\geq 0}(-1)^k c_k t^{\alpha k}. \tag{24}$$

with

$$c_0 = \frac{\Gamma(\alpha)^2}{\Gamma(2\alpha-1)\Gamma(2-\alpha)}\frac{\mu(0)}{\mu_\infty}, \quad \textit{and for every } k \geq 1, \ c_k \textit{ is defined inductively by:}$$

$$c_k = \frac{\Gamma(\alpha)^2 B(\alpha(k+1),\, 2(1-\alpha))}{\Gamma(2(1-\alpha))\,\Gamma(2\alpha-1)}\left[(a*b)_k - \alpha(k+1)\sum_{\ell=1}^{k} B(\alpha(\ell+2)-1,\, \alpha(k-\ell-1)+2)\,(b^{*2})_\ell\, c_{k-\ell}\right].$$

A Numerical illustration: (Fake) Stationary regime, α-fractional kernels $K_\alpha(t) = \frac{u^{\alpha-1}}{\Gamma(\alpha)}\mathbf{1}_{\mathbb{R}}(t)$:

Proposition (Existence and Properties of the function $\varsigma^2_{\alpha,\lambda,c}$ for $\alpha \in (\frac{1}{2}, \frac{3}{2})$)

The convergence radius of the fractional power series (24) *that defines* $\zeta_{\alpha,\lambda,c}$ *is infinite and* $\zeta_{\alpha,\lambda,c}$ *is positive on* $(0, +\infty]$ *so that* $\zeta_{\alpha,\lambda,c}$ *is well-defined: The stabilizer* $\zeta^2_{\alpha,\lambda,c}$ *exists as an non-negative function, such that:*

1. $\lim_{t\to 0} \zeta_{\alpha,\lambda,c} = \begin{cases} 0 \text{ if } \alpha \leq 1, \\ +\infty \text{ if } \alpha > 1, \end{cases}$ *and* $\lim_{t\to+\infty} \zeta_{\alpha,\lambda,c}(t) = \frac{\sqrt{c}\lambda}{\|f_{\alpha,\lambda}\|_{L^2(Leb_1)}}$.

We introduce an Euler-Maruyama scheme (25) on the time grid $t_k = t_k^n = \frac{kT}{n}, k = 0, \ldots, n$, for the semi-integrated form (22), which write recursively ($R_{\alpha,\lambda}(t) = 1 - \int_0^t f_{\alpha,\lambda}(s)\, ds$):

$$\overline{X}_{t_k} = \frac{\theta_0}{\lambda} + (X_0 - \frac{\theta_0}{\lambda})R_{\alpha,\lambda}(t_k) + \sum_{\ell=1}^{k} \frac{\nu}{\lambda} \int_{t_{\ell-1}}^{t_\ell} f_\lambda(t_k - s)\, \varsigma_{\alpha,\lambda,c}(t_\ell)\sqrt{\overline{X}_{t_{\ell-1}}}\, dW_s = g(t_k) + \frac{\nu}{\lambda}\sum_{\ell=1}^{k} \varsigma_{\alpha,\lambda,c}(t_\ell)\sqrt{\overline{X}_{t_{\ell-1}}}\, I_k^{n,l} \qquad (25)$$

where the integrals $\left(I_k^{n,l} = \int_{t_{\ell-1}}^{t_\ell} f_\lambda(t_k - s) dW_s\right)_k$ can be simulated on the discrete grid $(t_k^n)_{0\leq k\leq n}$ by generating an independent sequence of gaussian vectors $G^{n,l}, l = 1 \cdots n$ using the Cholesky decomposition of a well-defined covariance matrix C.

A Numerical illustration: Confluence of Stabilized Fractional-CIR Process. $\alpha \in (\frac{1}{2}, \frac{3}{2}) \subset (0, 2)$

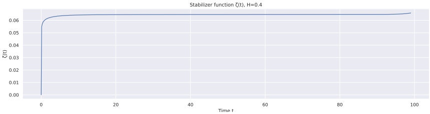


Figure: Graph of the stabilizer $t \rightarrow \varsigma_{\alpha,\lambda,c}(t)$ over time interval [0, T], T = 100 for a value of the Hurst esponent $H = 0.4$, $\lambda = 0.2$, c = 0.3.

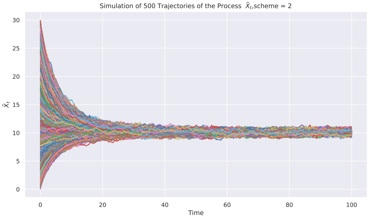


Figure: Confluence or Contraction from a [0,30]-Uniform Distribution.

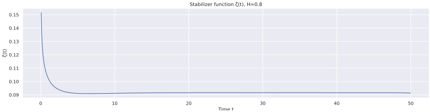
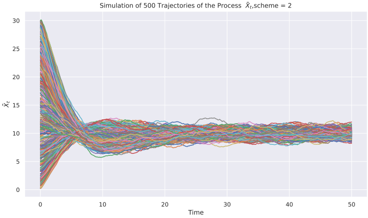


Figure: Graph of the stabilizer $t \rightarrow \varsigma_{\alpha,\lambda,c}(t)$ over time interval [0, T], T = 50 for a value of the Hurst esponent $H = 0.8$, $\lambda = 0.2$, c = 0.36.



A Numerical illustration: Fractional-CIR Process in the (Fake) Stationary regime with $\alpha \in (\frac{1}{2}, \frac{3}{2})$

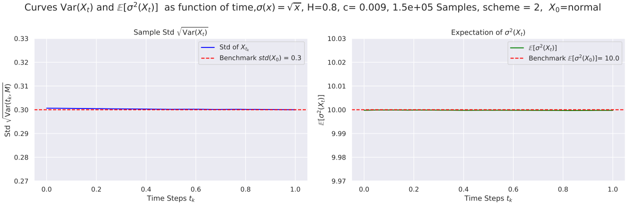
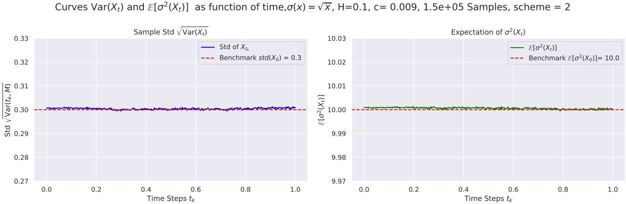


Figure: Graph of $t_k \mapsto \text{StdDev}(t_k, M)$ and $t_k \mapsto \mathbb{E}[\sigma^2(X_{t_k}, M)]$ over the time interval $[0, T]$, $T = 1$, $H = 0.8$, $\theta_0 = 2$, $\lambda = 0.2$, $v_0 = 0.09$, and $\text{StdDev}(X_0) = 0.3$, $\nu = 1$. Number of steps: $n = 800$, Simulation size: $M = 150000$.



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References/Citations

1. G. Pagès. Volterra equations with affine drift: looking for stationarity. Application to quadratic rough Heston model.
2. E. Gnabeyeu and G. Pagès. On a Stationarity Theory for Stochastic Volterra Integral Equations.
3. E. Gnabeyeu and G. Pagès and M. Rosenbaum. On Inhomogeneous Affine Volterra Processes: Stationarity and Applications to the Volterra Heston Model.
4. E. Gnabeyeu and G. Pagès and M. Rosenbaum. On the Microstructural Foundation of Inhomogeneous Rough Fractional Square Root Process, working paper.
5. R. Gorenflo and F. Mainardi. Fractional calculus: Integral and differential equations of fractional order.

Thanks For Your Attention!

Questions ?