On the fake stationary rough volatility models.

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LABORATOIRE DE PROBABILITÉS STATISTIQUE & MODÉLISATION

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The *short/rough memory* of *Volterra Integral* Diffusions
Università di Bologna - Bologna, Italia.

From Volterra Heston model to the fake stationary Volterra Heston model.

In the Volterra Heston model, the asset price process S and the variance process V are given by

$$\frac{dS_t}{S_t} = \sqrt{V_t}\,\big(\rho\, dW_t + \sqrt{1-\rho^2}\, dW_t^{\top}\big), \qquad S_0 \in (0,\infty), \tag{1}$$

$$V_t = \sigma^2(X_t), \quad X_t = X_0 + \int_0^t K_\alpha(t-s)\Big(\big(\mu(s) - \lambda X_s\big)\, ds + \nu\sqrt{V_s}\, dW_s\Big). \tag{2}$$

where K_α is the fractional kernel $\alpha := H + \frac{1}{2}$, $W = (W, W^{\top})$ is a two-dimensional independent standard Brownian motion $\rho \in [-1,1]$, μ a deterministic function, $\lambda, \nu \in \mathbb{R}_+$.

- If $\sigma(x) = \sqrt{x}$ and $\alpha \in (\frac{1}{2}, 1)$, we recover the standard rough heston model
- If $\sigma(x) = \sqrt{\kappa_0 + \kappa_2(x-a)^2}$ and $\alpha \in (\frac{1}{2}, 1)$, we recover the quadratic rough heston model

Potential challenges:

- Exhibits distinct regimes for short and long maturities
- Due to the intrinsic **non-stationarity** of the Volterra Equation (2), see, e.g. [Pagès, 2024]

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Solution:

- Introduce the **"fake stationary Volterra" Heston model** in the terminology of [Pagès, 2024]
- Ensures a time-consistent modeling framework

Fake stationary Volterra processes with affine drift and convolutive kernel

Fix $T > 0$, under the given complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P})$, let X be the following $\mathbb{R}$–valued scaled Volterra SDE driven by the 1-dimensional Brownian motion W:

$$X_t = X_0\varphi(t) + \int_0^t K(t-s)\big(\mu(s) - \lambda X_s\big)ds + \int_0^t K(t-s)\sigma(X_s)dW_s, \quad V_0 \perp\!\!\!\perp W. \tag{3}$$

Here, $K \in L^2([0,T],\mathbb{R})$ is a scalar kernel, φ a (locally) bounded Borel initial function and $\lambda \in \mathbb{R}_+$, $\mu : \mathbb{R}_+ \to \mathbb{R}$, $\sigma : \mathbb{R} \to \mathbb{R}$.

Assumption (Well-posedness): We assume there exits at least a continuous weak solution to the Volterra SDE (3).

Wiener–Hopf transform: Let f_λ be the resolvent of λK, (i.e. $f_\lambda + \lambda K * f_\lambda = \lambda K$), the solution $(X_t)_{t\ge 0}$ of the Volterra SDE (3) also satisfies:

$$X_t = X_0\Big(\phi(t) - \int_0^t f_\lambda(t-s)\phi(s)\,ds\Big) + \frac{1}{\lambda}\int_0^t f_\lambda(t-s)\mu(s)\,ds + \frac{1}{\lambda}\int_0^t f_\lambda(t-s)\sigma(X_s)\,dW_s. \tag{4}$$

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Definition 1 (Fake Stationarity)

The process $(X_t)_{t\geq 0}$ with diffusion coefficient $\sigma(X)$, starting from $X_0 \in L^2(\mathbb{P})$, exhibit a fake stationary regime of type I if:

$$\forall\, t \geq 0, \quad \mathbb{E}\,X_t = \mathbb{E}\,X_0 =: m_0, \quad \mathrm{Var}(X_t) = \mathrm{Var}(X_0) =: v_0 \quad \text{and} \quad \mathbb{E}\,\sigma^2(X_t) = \bar{\sigma}_0^2. \tag{5}$$

Theorem 2 (Autonomous diffusion coefficient σ [Gnabeyeu and Pagès, 2025])

Assume $X_0 \in L^2(\mathbb{P})$ and let $(X_t)_{t\geq 0}$ be a solution to the scaled Volterra Equation (3) starting from X_0 and satisfying

$$\forall\, t \geq 0, \quad \mathbb{E}\, X_t = m_0, \quad \mathrm{Var}(X_t) = v_0 \geq 0 \quad \text{and} \quad \mathbb{E}\,\sigma^2(X_t) = \bar{\sigma}_0^2 > 0. \tag{6}$$

Set $c := \frac{v_0}{\bar{\sigma}_0^2} > 0$ be such that $c \geq \frac{1}{\lambda^2}\int_0^\infty f_\lambda^2(s)\, ds$. Then, a necessary condition for the relations (6) to be satisfied is that

$$\forall\, t \geq 0, \quad \varphi(t) = 1 - \lambda \int_0^t K(t-s)\left(\frac{\mu(s)}{\lambda m_0} - 1\right) \mathrm{d}s. \tag{7}$$

and the mean-reverting function μ in Equation (3) is uniquely determined and is given by

$$\mu(t) := \frac{d}{dt}\Big(\int_0^t g_\lambda(t-s) r_\lambda(ds)\Big) \quad \text{with} \quad g_\lambda(t) = \lambda m_0\Big(1 - \sqrt{1 - \frac{1}{c\,\lambda^2}\int_0^t f_{\alpha,\lambda}^2(s)\, ds}\Big) \text{ and } (f_\lambda * r_\lambda)(t) = (r_\lambda * f_\lambda)(t) = 1 \tag{8}$$

provided the functional resolvent of the first kind r_λ of f_λ is well-defined, In such case, the process $(X_t)_{t\geq 0}$ has constant mean m_0 and a stationary variance v_0 over time.

Remark 1: In such case, the Volterra Equation (3) simplify to

$$X_t = X_0 - \Big(X_0 - m_0\Big)\Big(1 - \sqrt{1 - \frac{1}{c\,\lambda^2}\int_0^t f_\lambda^2(s)\,ds}\Big) + \frac{1}{\lambda}\int_0^t f_\lambda(t-s)\sigma(X_s)dW_s. \tag{9}$$

Example 3 (The case of α-fractional integration kernels $K_\alpha(t) = \frac{u^{\alpha-1}}{\Gamma(\alpha)}\mathbf{1}_{\mathbb{R}_+}(t)$)

Let $\alpha \in (0,1)$ and consider the fractional integration kernel $K_\alpha(t) = \frac{u^{\alpha-1}}{\Gamma(\alpha)}\mathbf{1}_{\mathbb{R}_+}(t)$, with $f_{\alpha,\lambda}$ defined as the resolvent of $\lambda\,K_\alpha$.
Let $c \geq \frac{1}{\lambda^2}\int_0^\infty f_{\alpha,\lambda}^2(s)\,ds$. In view of Equation (8), it follows that:

$$\mu(t) = g_{\alpha,\lambda}(t) + \frac{1}{\lambda}D^\alpha g_{\alpha,\lambda}(t) \quad \text{with} \quad g_{\alpha,\lambda}(t) = \lambda m_0\Big(1 - \sqrt{1 - \frac{1}{c\,\lambda^2}\int_0^t f_{\alpha,\lambda}^2(s)\,ds}\Big)$$

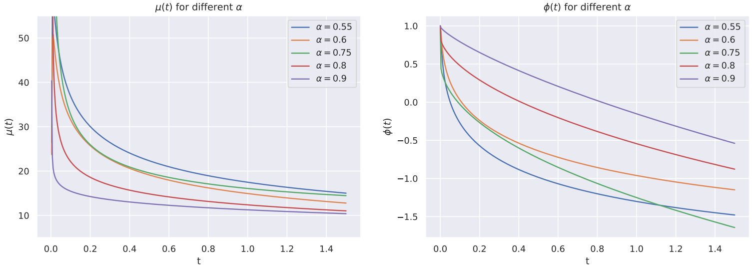
and $D^\alpha := \frac{d}{dt}I^{1-\alpha}$ where D^α and $I^{1-\alpha}$ denote, respectively, the Riemann–Liouville fractional derivative of order α, and the Riemann–Liouville fractional integral of order $1-\alpha$

Remark 2: One checks that μ is continuous on $(0,+\infty)$ and that $\lim_{t\to+\infty}\mu(t) \in \mathbb{R}$. Moreover, by Hardy-Littlewood's Tauberian theorem for Laplace transform, we have (heuristically), $\mu(t) \underset{0^+}{\sim} \frac{m_0\,\Gamma(2\alpha)}{2c(2\alpha-1)\Gamma(\alpha)^3}\,t^{\alpha-1}$.

Potential challenges:

- Highly restrictive, as it effectively dictates explicit closed-form for both ϕ and the mean-reversion term μ in the drift.

A numerical illustration of intrinsic fake stationary



Curves $\mathbb{E}[X_t]$ and $\mathrm{Var}(X_t)$ as function of time, $\sigma(x) = \sqrt{\kappa_0 + \kappa_2(x - m_0)^2}$, H=0.4 , 1.0e+05 Samples, $X_0 \sim \mathcal{N}(m_0, v_0)$

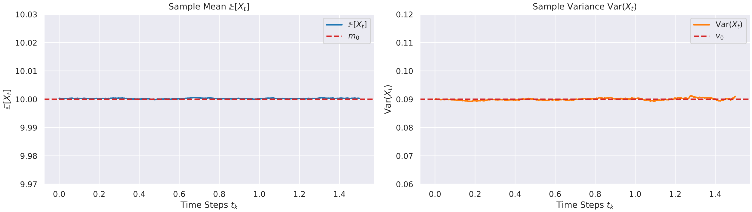


Figure: *Top: Graphs of* $t \mapsto \mu(t)$ *and* $t \mapsto \varphi(t)$ *for different values of* α. *Bottom: Graphs of* $t_k \mapsto \mathbb{E}[X_{t_k}]$ *and* $t_k \mapsto \mathrm{Var}(X_{t_k}, M)$

Fake stationary Volterra process using a stabilizer

$$X_t = X_0\varphi(t) + \int_0^t K(t-s)\big(\underbrace{\mu(s)}_{!!} - \lambda X_s\big)ds + \int_0^t K(t-s)\varsigma(s)\sigma(X_s)dW_s, \quad V_0 \perp\!\!\!\perp W. \tag{10}$$

Theorem 4 (Time-dependent diffusion coefficient [Gnabeyeu and Pagès, 2025].)

Let $(X_t)_{t\geq 0}$ be a solution to the scaled Volterra equation in its form (3) starting from any random variable $X_0 \in L^2(\Omega, \mathcal{F}, \mathbb{P})$. Assume $X_0 \in L^2(\mathbb{P})$ and let $(X_t)_{t\geq 0}$ be a solution to (3) satisfying

$$\forall\, t \geq 0, \quad \mathbb{E}\, X_t = m_0 \neq 0, \quad \mathrm{Var}(X_t) = v_0 \geq 0 \quad \textit{and} \quad \bar{\sigma}(t) := \mathbb{E}\,\sigma^2(X_t) = \bar{\sigma}_0^2 > 0. \tag{11}$$

Then, a necessary and sufficient condition for the relations (11) to be satisfied is that

$$\forall\, t \geq 0, \quad \varphi(t) = 1 - \lambda \int_0^t K(t-s)\left(\frac{\mu(s)}{\lambda m_0} - 1\right) \mathrm{d}s. \tag{12}$$

and the couple $(v_0, \varsigma(t))$ must satisfy the functional equation:

$$(E_{\lambda,c}): \quad \forall\, t \geq 0, \quad c\lambda^2\Big(1 - \big(1 - \frac{(f_\lambda * \mu)_t}{\lambda m_0}\big)^2\Big) = (f_\lambda^2 * \varsigma^2)(t) \quad \textit{with} \quad c = \frac{v_0}{\bar{\sigma}_0^2} \quad \textit{and} \quad \varsigma := \varsigma_{\lambda,c}. \tag{13}$$

Definition 5 (Stabilizer)

We will call the stabilizer (or corrector) of the scaled stochastic Volterra equation (3) the (locally) bounded Borel function $\varsigma := \varsigma_{\lambda,c}$, solution (if any) to the functional equation $(E_{\lambda,c})$ in (13).

- The computation of the function $\varsigma^2_{\alpha,\rho,\lambda,c}$, involves knowing the *form of the mean-reverting function* μ

We can solve the functional equation numerically

Example 6 (Computing $\varsigma^2_{\alpha,\rho,\lambda,c}$ with second-order extension of the triangular Volterra discretization)

On a time grid $t_k = k\frac{T}{n}, k = 0, ..., n.$, using a piecewise linear interpolation on $[t_j, t_{j+1}]$, we build the recursive scheme: $\varsigma^2_{\alpha,\rho,\lambda,c}(0) = 0$ and $\forall\, k \geq 1$,

$$c\lambda^2\Big(1-\big(1 - \frac{(f_\lambda * \mu)((t_k))}{\lambda m_0}\big)^2\Big) = (f^2_{\alpha,\rho,\lambda} * \varsigma^2_{\alpha,\rho,\lambda,c})(t_k) \tag{14}$$

$$= \sum_{j=0}^{k-1}\Big(\varsigma^2_{\alpha,\rho,\lambda,c}(t_j)\int_{t_j}^{t_{j+1}} f^2_{\alpha,\rho,\lambda}(t_k - s)\frac{t_{j+1}-s}{t_{j+1}-t_j}\,ds + \varsigma^2_{\alpha,\rho,\lambda,c}(t_{j+1})\int_{t_j}^{t_{j+1}} f^2_{\alpha,\rho,\lambda}(t_k - s)\frac{s-t_j}{t_{j+1}-t_j}\,ds\Big).$$

which we can solve step by step (Lower-Triangular system) to recover the values $\varsigma^2_{\alpha,\rho,\lambda,c}(t_k)$, $k \geq 1$.

Remark: In this setting, the Volterra SDE (3) simplify to

$$X_t = X_0 - \frac{1}{\lambda m_0}\Big(X_0 - m_0\Big)\int_0^t f_\lambda(t-s)\mu(s)\,\mathrm{d}s + \frac{1}{\lambda}\int_0^t f_\lambda(t-s)\varsigma_{\alpha,\rho,\lambda,c}(s)\sigma(X_s)dW_s. \tag{15}$$

Illustrations with $\alpha-$ exponential fractional kernels $K_{\alpha,\rho}(t) = e^{-\rho t}\frac{u^{\alpha-1}}{\Gamma(\alpha)}\mathbf{1}_{\mathbb{R}}(t)$, $\rho > 0, \alpha \in (\frac{1}{2}, 1)$

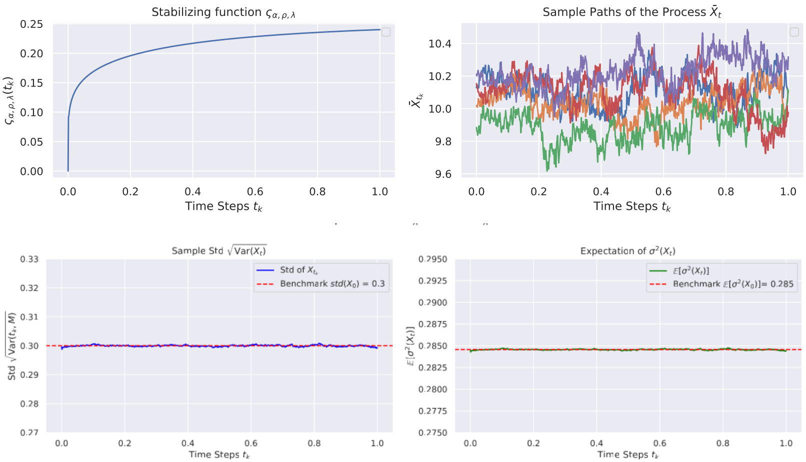


Figure: *Stabilizer, sample paths, empirical variance, and mean squared volatility of* $(X_t)_{t\in[0,1]}$ *with parameters:* $\sigma(x) = \sqrt{\kappa_0 + \kappa_2(x - m_0)^2}$, $\mu(t) = \lambda m_0$, $H = 0.1$, $c = 0.316$, $\lambda = 0.2$, $\rho = 1.2$, $m_0 = 10$, $v_0 = 0.09$, $\kappa_0 = 0.25$ *and* $\kappa_2 = 0.384$.

Applications I: The fake stationary rough Heston model, $K(t) = K_\alpha(t) = \frac{u^{\alpha-1}}{\Gamma(\alpha)}\mathbf{1}_{\mathbb{R}}(t)$,$\alpha \in (\frac{1}{2}, 1)$

In this setting, the pair $(\log S, V)$, where S denotes the asset price and V its variance process, is governed by

$$\begin{cases} \frac{dS_t}{S_t} &= \sqrt{V_t}\Big(\rho\, dW_t + \sqrt{1-\rho^2}\, dW_t^\top\Big), \qquad S_0 \in (0,\infty), \\ V_t &= V_0 + \int_0^t K_\alpha(t-s)\Big(\big(1-\frac{V_0}{m_0}\big)\mu(s) - \lambda\big(V_s - V_0\big)\Big)\,\mathrm{d}s + \int_0^t K_\alpha(t-s)\nu\varsigma_{\alpha,\lambda,c}(s)\sqrt{V_s}dW_s. \end{cases} \tag{16}$$

with:

$$\forall\, t \geq 0, \quad c\lambda^2\Big(1 - \big(1 - \frac{(f_{\alpha,\lambda} * \mu)_t}{\lambda m_0}\big)^2\Big) = (f^2_{\alpha,\lambda} * \varsigma^2_{\alpha,\lambda,c})(t) \quad with \quad c, \lambda > 0. \tag{17}$$

where, $W = (W, W^\top)$ is a two-dimensional standard Brownian motion, $\rho \in [-1,1]$ a correlation, μ a deterministic function, $\lambda, \nu, c \in \mathbb{R}_+$ such that V is at least a weak solution to the Volterra equation (16) which writes equivalently:

$$V_t = V_0 + \frac{1}{\lambda}\int_0^t f_{\alpha,\lambda}(t-s)\big(1 - \frac{V_0}{m_0}\big)\mu(s)ds + \frac{1}{\lambda}\int_0^t f_{\alpha,\lambda}(t-s)\nu\varsigma_{\alpha,\lambda,c}(s)\sqrt{V_s}dW_s, \quad \varsigma_{\alpha,\lambda,c}(t), \geq 0 \tag{18}$$

Well-Posedness: Unique-in-law positive continuous weak solution (as a scaling limit of a sequence of time-modulated Hawkes processes) see e.g., [Gnabeyeu et al., 2025].

Applications I: The fake stationary rough Heston model, $K(t) = K_\alpha(t) = \frac{u^{\alpha-1}}{\Gamma(\alpha)}\mathbf{1}_{\mathbb{R}}(t), \alpha \in (\frac{1}{2}, 1)$

Assumption 7 (Integrability and uniform Hölder continuity)

Let $\lambda > 0$ and $\alpha \in (\frac{1}{2}, 1)$. Assume the kernel K_α is such that its resolvent $f_{\alpha,\lambda}$ satisfies:

(i) Integrability: $\int_0^{+\infty} f_{\alpha,\lambda}^{2\beta}(u)\, du < +\infty$ *for some* $\beta > 1$.

(ii) Hölder continuity: $\exists\, \vartheta \in (0, 1]$ *and* $C < +\infty$ *such that* $\max_{i=1,2} \left[\int_0^{+\infty} |f_{\alpha,\lambda}(u+\bar{\delta}) - f_{\alpha,\lambda}(u)|^i\, du\right]^{\frac{1}{i}} \le C\bar{\delta}^{\vartheta}$.

Theorem 8 ([Gnabeyeu et al., 2026])

The fake stationary rough Heston model (16) *has a unique in law continuous* $\mathbb{R}_+ \times \mathbb{R}_+$*-valued weak solution* (S, V) *for any initial condition* $(S_0, V_0) \in \mathbb{R}_+ \times L^2_{\mathbb{R}_+}(\mathbb{P})$ *defined on some filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F})_{t\ge 0}, \mathbb{P})$*. Moreover, Let* $p > 0$ *such that* $V_0 \in L^p(\mathbb{P})$*. Under Assumption 7, the sample paths of* V *are* $\left(\vartheta \wedge \frac{\beta-1}{2\beta} - \frac{1}{p} - \eta\right)$*-Hölder pathwise continuous for sufficiently small* $\eta > 0$ *and for any* $T > 0$*, there exists a constant* $C_{p,T} > 0$ *such that the following sup-norm holds*

$$\Big\| \sup_{t\in[0,T]} |V_t| \Big\|_p^p = \mathbb{E}\Big[\sup_{t\in[0,T]} |V_t|^p\Big] \le C_{p,T}\left(1 + \mathbb{E}[|V_0|^p]\right). \tag{19}$$

Sketch of Proof: The uniqueness in law follows from the characteristic function. Since S is fully determined by V, the existence of S readily follows from that of V.

Asymptotic Hawkes Setting: Let $(N_t^T, \Lambda_t^T)_{t\geq 0}$ be a sequence of 1-dimensional Hawkes processes indexed by $T > 0$, observed over $[0, T]$, and defined on the filtered probability space $(\Omega^T, \mathcal{F}^T, \mathbb{P}^T, (\mathcal{F}_t^T)_{t\in[0,T]})$ with $T \to \infty$ and the intensity process $(\Lambda_t^T)_{t\geq 0}$ given by:

$$\Lambda_t^T = \underbrace{\Lambda_0^{*T}\Phi^T(t)}_{!!} + \mu^T(t) + \int_0^t \varphi^T(t-s)\, dN_s^T \tag{20}$$

where:

- $\mu^T(t)$ is the non-negative baseline intensity.
- φ^T is a non-negative measurable and completely monotonic memory kernel with regular varying tail and $\|\varphi^T\|_1 < \infty$,
- $\Phi^T(t) = \int_t^\infty \varphi^T(s)\, ds$ is the non-negative initial condition.
- Λ_0^{*T} a positive real-valued random variable

Ideas

Introduce a time-dependent rescaling. Define $\Lambda_t^{*T} := \frac{1-a_T}{\tilde{\mu}^T(\frac{t}{T})}\Lambda_{tT}^T$, where $(a_T)_{T\geq 0}$ is a $(0,1)$-valued sequence of positive numbers converging to 1.

Let Ψ^T be the resolvent of φ^T. Wiener–Hopf transform of the intensity process (Λ_t^{*T}) and asymptotic limit as $T \to +\infty$.

$$\Lambda_t^{*T} = \underbrace{\Lambda_0^{*T}\left(\Phi^T(tT) + (\Psi^T * \Phi^T)(tT)\right)\frac{1-a_T}{\tilde{\mu}^T(\frac{t}{T})}}_{\underset{T\to+\infty}{\to} \Lambda_0^*\left(\phi(t) - \int_0^t f_{\alpha,\lambda}(t-s)\phi(s)ds\right)} + \frac{\mu^T}{\tilde{\mu}^T(\frac{t}{T})}\int_0^t \underbrace{T(1-a_T)\Psi^T(T(t-s))}_{\underset{T\to+\infty}{\to} f_{\alpha,\lambda}(t-s)} \underbrace{\frac{\mu^T(Ts)}{\mu^T}}_{\underset{T\to+\infty}{\to} \frac{1}{\lambda}\mu(s)} ds$$

$$+ \underbrace{(1-a_T)\frac{\mu^T(tT)}{\tilde{\mu}^T(\frac{t}{T})}}_{\underset{T\to+\infty}{\to} 0} + \int_0^t \underbrace{\sqrt{\frac{\tilde{\mu}^T(\frac{s}{T})}{T(1-a_T)(\tilde{\mu}^T(\frac{t}{T}))^2}}}_{\underset{T\to+\infty}{\to} \frac{\nu}{\lambda}\varsigma(s)} \underbrace{T(1-a_T)\Psi^T(T(t-s))}_{\underset{T\to+\infty}{\to} f_{\alpha,\lambda}(t-s)} \sqrt{\Lambda_s^{*T}}dW_s^T, \ \forall t \in [0, t_0]$$

with some constant $\lambda, \nu \in \mathbb{R}_+^*$ and some deterministic and continuous positive function ϕ, μ and ς.
Taking the limit as $T \to \infty$, we expect the limiting process Λ_t^* to be the solution (with some additional assumptions) of

$$\Lambda_t^* = \Lambda_0^*\left(\phi(t) - \int_0^t f_{\alpha,\lambda}(t-s)\phi(s)ds\right) + \frac{1}{\lambda}\int_0^t f_{\alpha,\lambda}(t-s)\mu(s)ds + \frac{1}{\lambda}\int_0^t f_{\alpha,\lambda}(t-s)\varsigma(s)\sqrt{\Lambda_s^*}dW_s. \tag{21}$$

or equivalently (Wiener–Hopf transform),

$$\Lambda_t^* = \Lambda_0^*\phi(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\mu(s) - \lambda\Lambda_s^*)\,ds + \frac{\lambda\nu}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varsigma(s)\sqrt{\Lambda_s^*}\,dW_s. \tag{22}$$

Formal Proof: By Functional Limits Theorem: "À la Jacod–Shiryaev", see [Gnabeyeu et al., 2026]

Applications I: The characteristic function of the fake stationary rough Heston Model.

Theorem 9 (Fourier–Laplace Transform of the fake stationary rough Heston model [Gnabeyeu et al., 2025])

Let $\alpha \in (\frac{1}{2}, 1)$, $\lambda, c > 0$ and $u \in (\mathbb{C}^2)^*$. Then there exists a unique function $\psi \in L^2([0,T], \mathbb{C})$ solving the fractional Riccati equation

$$(D^\alpha \psi)(t) = \frac{1}{2}(-u^2 - iu) + \left(iu\rho\nu\, \varsigma_{\alpha,\lambda,c}(T-t) - \lambda\right)\psi(t) + \frac{\nu^2}{2}\varsigma^2_{\alpha,\lambda,c}(T-t)\,\psi^2(t), \quad t \in [0,T], \quad (I^{1-\alpha}\psi)(0) = 0. \tag{23}$$

leading to the Fourier–Laplace representation for the log-price:

$$\varphi_{T,V_0}(u) = \mathbb{E}_{V_0}\left[e^{iu\log(\frac{S_T}{S_0})}\right] = \exp\left(V_0(I^{1-\alpha}\psi)(T) + \int_0^T \psi(T-s)\left(\left(1 - \frac{V_0}{m_0}\right)\mu(s) + \lambda V_0\right)\mathrm{d}s\right). \tag{24}$$

with $D^\alpha = \frac{d}{dt}I^{1-\alpha}$ where D^α and $I^{1-\alpha}$ denote, respectively, the Riemann–Liouville fractional derivative of order α, and the Riemann–Liouville fractional integral of order $1-\alpha$.

Practitionner corner:

- Generalized Adams–Bashforth–Moulton algorithm (fractional Adams method) for solving a fractional ODE (23).
- Plugging the numerical solution into (24) yields the characteristic function, from which standard Fourier methods allow the pricing of call and put options (Carr Madan, Lewis, etc.).

The Quadratic Rough Heston volatility Model, $K_\alpha(t) = \frac{u^{\alpha-1}}{\Gamma(\alpha)}\mathbf{1}_{\mathbb{R}}(t)$, $\alpha \in (\frac{1}{2}, 1)$ and $\alpha := H + \frac{1}{2}$

- has been introduce for the joint calibration of the S&P 500 and VIX volatility surfaces, accounting for the so-called Zumbach effect.
- typically used to model options with short expiries.

Let $(X_t)_{t\geq 0}$ be a solution to the fake stationary quadratic Volterra SDE, where for every $t > 0$, $\underline{\mu(t) = \lambda m_0}$, so that $\varphi(t) = 1$, $\forall t \geq 0$. We define the model

$$\begin{cases} \dfrac{dS_t}{S_t} = \sigma(X_t)\Big(\rho\, dW_t + \sqrt{1-\rho^2}\, dW_t^\top\Big), \\ X_t = g_0(t) + \dfrac{1}{\lambda}\displaystyle\int_0^t f_{\alpha,\lambda}(t-s)\,\varsigma_{\alpha,\lambda,c}(s)\,\sigma(X_s)\,dW_s, \end{cases} \quad \text{with} \quad \begin{cases} \sigma(x) = \sqrt{\kappa_0 + \kappa_2(x-m_0)^2}, \qquad \kappa_i \geq 0,\ i = 0, 2, \\ g_0(t) = m_0 + (X_0 - m_0)\Big(1 - \displaystyle\int_0^t f_{\alpha,\lambda}(s)\,ds\Big), \end{cases}$$

and $\varsigma_{\alpha,\lambda,c}(t) \geq 0$, solution to

$$(E_{\alpha,\lambda,c}): \quad \forall\, t \geq 0, \quad c\lambda^2\Big(1 - \big(1 - \int_0^t f_{\alpha,\lambda}(s)\,ds\big)^2\Big) = (f^2_{\alpha,\lambda} * \varsigma^2_{\alpha,\lambda,c})(t) \quad \textit{with} \quad \lambda > 0 \text{ and } c \in \Big(0, \frac{1}{\kappa_2}\Big). \tag{25}$$

$$X_0 \sim \mathcal{L}\Big(m_0, v_0 = \frac{c\kappa_0}{1 - c\kappa_2}\Big), \quad \underline{\textit{Model Parameters:}} \quad \vec{\theta} := (\alpha, \lambda, \rho, c, m_0, \kappa_0, \kappa_2)$$

- $\Gamma(a) = \int_0^{+\infty} u^{a-1}e^{-u}\,du,\ a>0$ and $B(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}\,du,\ a,b>0$, the Gamma and the beta functions.

Set $a_k = \frac{1}{\Gamma(\alpha k+1)}, b_k = \frac{1}{\Gamma(\alpha(k+1))},\ k\geq 0$. The solution $\varsigma^2_{\alpha,\lambda,c}$ to $(E_{\alpha,\lambda,c})$ reads:

Example 10 (Stabilizer $\varsigma^2_{\alpha,\lambda,c}$, $\alpha\in(\frac{1}{2},1)$, see [Pagès, 2024])

$$\varsigma^2_{\alpha,\lambda,c}(t) = c\lambda^{2-\frac{1}{\alpha}}\varsigma^2_\alpha(\lambda^{\frac{1}{\alpha}}t) \quad \text{where} \quad \varsigma^2_\alpha(t) := 2\,t^{1-\alpha}\sum_{k\geq 0}(-1)^k c_k t^{\alpha k}. \tag{26}$$

with

$$c_0 = \frac{\Gamma(\alpha)^2}{\Gamma(2\alpha-1)\Gamma(2-\alpha)},$$ and for every $k\geq 1$, c_k is defined inductively by:

$$c_k = \frac{\Gamma(\alpha)^2 B(\alpha(k+1),\,2(1-\alpha))}{\Gamma(2(1-\alpha))\,\Gamma(2\alpha-1)}\left[(a*b)_k - \alpha(k+1)\sum_{\ell=1}^{k} B(\alpha(\ell+2)-1,\,\alpha(k-\ell-1)+2)\,(b^{*2})_\ell\, c_{k-\ell}\right].$$

where for two sequences of real numbers $(u_k)_{k\geq 0}$ and $(v_k)_{k\geq 0}$, the Cauchy product is defined as $(u*v)_k = \sum_{\ell=0}^k u_\ell v_{k-\ell}$

<u>*Sketch of Proof:*</u> Regular Variation (Tauberian theorems) on Laplace transforms of $(E_{\alpha,\lambda,c})$, see again [Pagès, 2024].

Practitioner's corner: Truncating the power series defining $\varsigma^2_{\alpha,\lambda,c}$ at $n_{\max} = 50$

The left panel displays $\log_{10}|c_k|$ as a function of k for $\alpha \in [0.51, 0.99]$. All curves fall below the float64 machine precision threshold, well before $k = 50 \Rightarrow$ Truncating the series (26) at $n_{\max} = 50$ is sufficient across the entire simulation range.

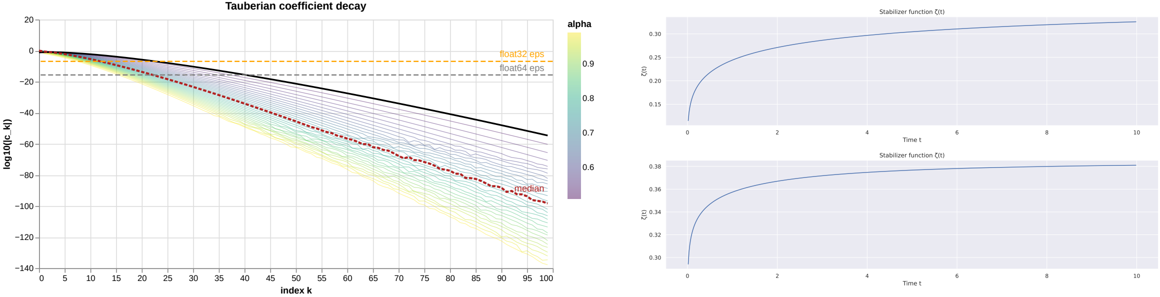


Figure: *Left: Tauberian series coefficients over the time interval* $[0, 10]$ *for Hurst exponents* $H \in (0, \frac{1}{2})$. *Right: Stabilizer mappings* $t_k \mapsto \varsigma_{\alpha,\lambda,c}(t_k)$ *over the time interval* $[0, 10]$ *for* $H = 0.1$ *and* $H = 0.4$. *Parameters* $\lambda = c = 1$ *and* $n = 600$.

The right panel depicts the stabilizing functions $t \mapsto \varsigma_{\alpha,\lambda,c}(t)$ for different values of $\alpha \in (\frac{1}{2}, 1)$.

On the discrete grid $(t_k^n)_{0\le k\le n}$ $t_k^n = \frac{kT}{n}$, $k = 0, \dots, n$,

- for the Markovian log-stock price process $\log S$, we use the Euler–Maruyama scheme

$$\log S_{t_k^n} = \log S_{t_{k-1}} - \frac{1}{2}\frac{V_{t_{k-1}} + V_{t_k^n}}{2}\frac{T}{n} + \sqrt{V_{t_{k-1}}}\left(\rho\,\Delta W_{t_k^n} + \sqrt{1-\rho^2}\,\Delta W^{\perp}_{t_k^n}\right), \tag{27}$$

where $\Delta W^{\perp}_{t_k^n} \sim \mathcal{N}(0, \frac{T}{n})$ is independent of all other sources of randomness.

- for the Variance process, we use the semi-integrated Euler scheme which write recursively:

$$\overline{X}^n_{t_k} = g_0(t_k) + \frac{1}{\lambda}\sum_{\ell=0}^{k-1} \varsigma_{\alpha,\lambda,c}(t_{\ell+1})\sigma(\overline{X}^n_{t_\ell})\int_{t_\ell}^{t_{\ell+1}} f_{\alpha,\lambda}(t_k^n - s)\,dW_s \qquad k = 1, \dots, n. \tag{28}$$

Let us denote by $G = (G_{k\ell})_{k=0:n,\ell=0:n-1}$ the $(n+1)\times n$ matrix involving the random terms $I_k^{n,\ell} := \int_{t_\ell}^{t_{\ell+1}} f_{\alpha,\lambda}(t_k^n - s)\,dW_s$.

$$(G_{k\ell})_{k=0:n,\ell=0:n-1} = \begin{pmatrix} \Delta W_{t_\ell}\mathbf{1}_{\{k=\ell\}},\ k\le \ell,\ \ell = 0:n-1 \\ \int_{t_\ell}^{t_{\ell+1}} f_{\alpha,\lambda}(t_k - s)\,dW_s\mathbf{1}_{\{0\le\ell<k\le n\}} \end{pmatrix}. \tag{29}$$

Then the following relation holds: $\overline{X}^n_0 = X_0$ and for every $k = 1, \dots, n$

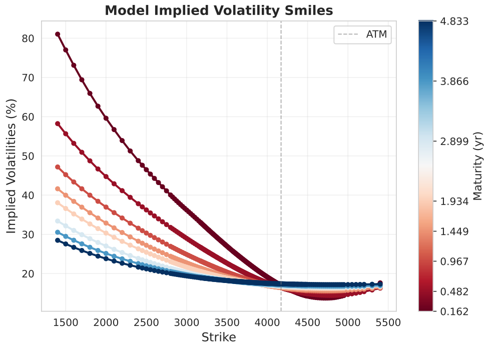
$$\left(\overline{X}^n_{t_k}\right)_{k=1:n} = \left(g_0(t_k)\right)_{k=1:n} + G_{k,\cdot}\left(\varsigma_{\alpha,\lambda,c}(t_{\ell+1})\sigma(\overline{X}^n_{t_\ell})\right)_{\ell=0:n-1} \tag{30}$$

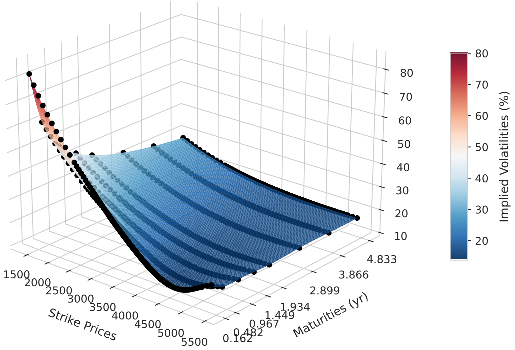
by generating an independent sequence of gaussian vectors $G_{k,\cdot}$, $k = 0\cdots n$ using an alternate extended and stable version of Cholesky decomposition of a well-defined covariance matrix, see [Gnabeyeu and Pagès, 2025, Gnabeyeu and Pagès, 2026].

Implied volatility surface generated by the FS-QRH model

Parameters	$\alpha = H + \frac{1}{2}$	λ	ρ	c	m_0	κ_0	κ_2
Fake stationary QRH	0.7478	1.4810	0.8575	0.2805	0.0752	0.0114	3.5652

Table: *Parameters for the Fake Stationary Quadratic Rough Heston Model*





- The model reproduces the patterns of differents expiries, producing a fairly good representation of a market vol. surf.

- We are left to calibrate our set of parameters $\vec{\theta} := (\alpha, \lambda, \rho, c, m_0, \kappa_0, \kappa_2)$: (A very small number of parameters).

$\Rightarrow$ Choice of the target objective function:[1] : Among Others, the weighted squared-error loss

$$F(\vec{\theta}) = \frac{1}{\#\sigma^{SPX}} \sum_{o \in \sigma^{SPX}} w^{o}_{\mathrm{vol}} \big(\sigma^{o,\mathrm{mid}} - \Psi^{o,vol}_{\vec{\theta}}\big)^2, \quad \text{with} \quad w^{o}_{\mathrm{vol}} := \frac{1}{\varepsilon + \left|\sigma^{o,\mathrm{bid}} - \sigma^{o,\mathrm{ask}}\right|}.$$

Here, σ^{SPX} the given set of SPX options, $\sigma^{o,\mathrm{mid}}$ the market mid implied volatility for the option o, and $\Psi^{o,vol}_{\vec{\theta}}$ is the implied volatility of the option o in the FS-QRH model with the given parameter $\vec{\theta}$ and obtained by Monte-Carlo simulations.

Our method of simulation (the semi-integrated Euler scheme) exhibit several advantages:

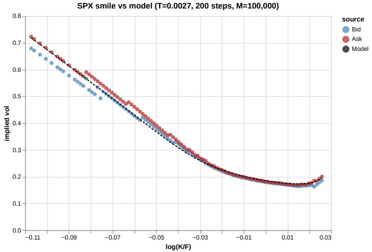
- its generality (even for long memory kernels),
- accuracy and computational efficiency

[1] Practitioners aim at positioning the model implied volatility at the mid.

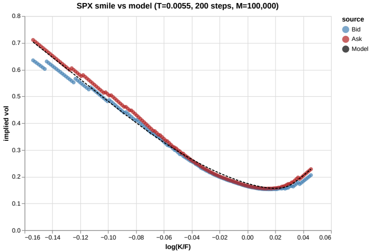
Model fit for very Short-Dated-Options (expiries measured in days)

Expiries	$\alpha = H + \frac{1}{2}$	λ	ρ	c	m_0	κ_0	κ_2
$T = 1$	0.5642	4.7711	0.7796	0.2890	0.0481	0.0218	3.4598
$T = 2$	0.6168	4.7430	0.8224	0.2835	0.0517	0.0171	3.5273
$T = 6$	0.6524	2.2821	0.8986	1.0814	0.0729	0.0115	0.9247

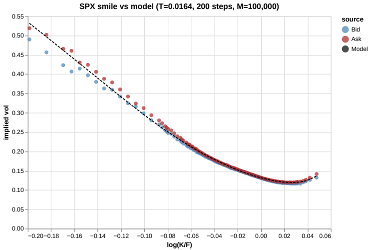
Table: *Calibrated parameters by expiry for three very short-dated options*



T=1 day, $F = 4146.74$, 101 strikes



T=2 days, $F = 4146.46$, 158 strikes



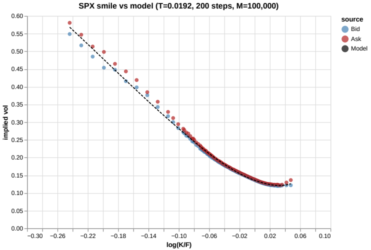
T=6 days, $F = 4147.00$, 109 strikes

Figure: *Market bid–ask implied volatilities and fitted model smile for three very short SPX maturities.*

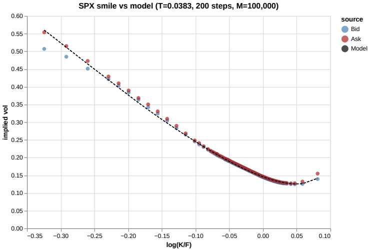
Model fit for Short-Dated-Options (expiries measured in weeks)

Expiries	$\alpha = H + \frac{1}{2}$	λ	ρ	c	m_0	κ_0	κ_2
$T = 1$	0.6138	3.2444	0.6877	0.7978	0.0747	0.0094	1.2535
$T = 2$	0.5561	2.9191	0.5966	0.2173	0.0634	0.0031	4.6030
$T = 3$	0.6144	1.7727	0.5812	0.2911	0.0864	0.0011	3.4354

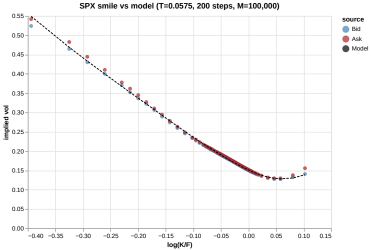
Table: *Calibrated parameters by expiry for three short-dated-options.*



T=1 week, $F = 4147.57$, 110 strikes



T=2 weeks, $F = 4149.69$, 103 strikes

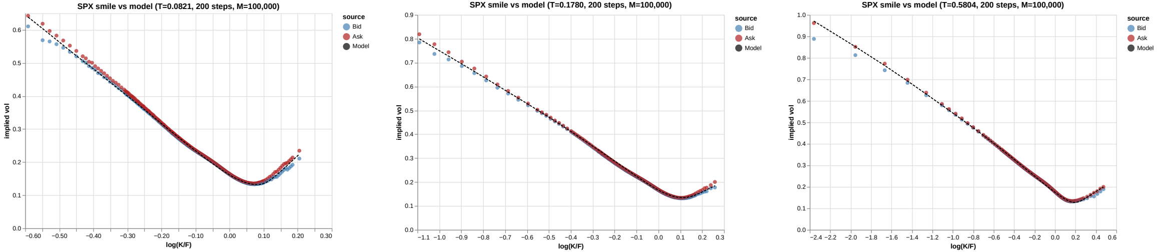


T=3 weeks, $F = 4152.24$, 89 strikes

Figure: *Market bid–ask implied volatilities and fitted model smile for three short SPX maturities.*

Expiries	$\alpha = H + \frac{1}{2}$	λ	ρ	c	m_0	κ_0	κ_2
$T = 1$	0.6386	1.4906	0.8717	0.5330	0.0921	0.0108	1.8762
$T = 2.17$	0.5929	1.3453	0.8985	0.5784	0.1139	0.0077	1.7289
$T = 6.1$	0.5894	1.6437	0.8909	0.2115	0.1002	0.0026	4.7277

Table: *Calibrated parameters for three medium term expiries .*



$T = 1$, $F = 4155.67$, 299 strikes T=2.17, $F = 4170.88$, 280 strikes $T = 6.1$, $F = 4233.80$ 134 strikes

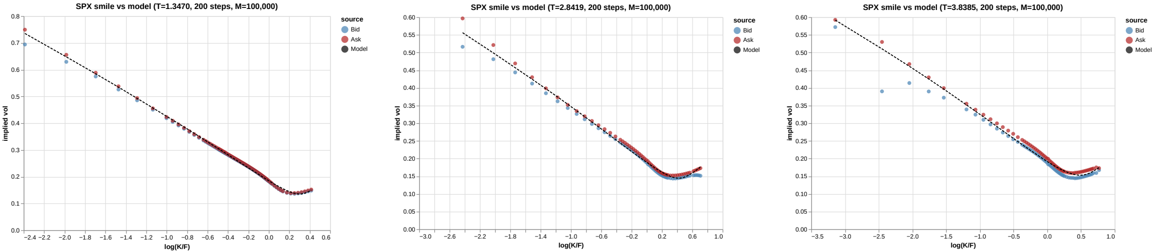
Figure: *Market bid–ask implied volatilities and fitted model smile for three medium term SPX maturities.*

- The fits are almost perfect on the three expiries,
- the model prices lies between the bid-ask spreads despite market tightness.

Model fit for Long-Dated Options (expiries measured in years)

Expiries	$\alpha = H + \frac{1}{2}$	λ	ρ	c	m_0	κ_0	κ_2
$T = 1.35$	0.5838	0.9219	0.8796	1.7510	0.3166	0.0016	0.5711
$T = 2.842$	0.5656	0.7569	0.8869	0.3611	0.1573	0.0003	2.7695
$T = 3.838$	0.5646	0.7736	0.8444	1.0135	0.2768	0.0003	0.9867

Table: *Calibrated parameters for three long-dated expiries.*



$T = 1.35,\ F = 4357.90,\ 105$ strikes $\quad$ $T = 2.842,\ F = 4535.03,\ 58$ strikes $\quad$ $T = 3.838,\ F = 4643.29,\ 61$ strikes

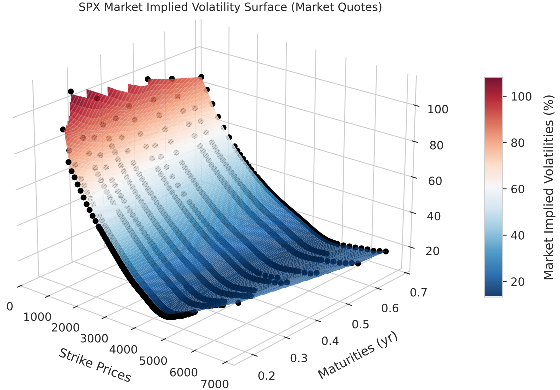
Figure: *Market bid–ask implied volatilities and fitted model smile for three long-dated SPX maturities.*

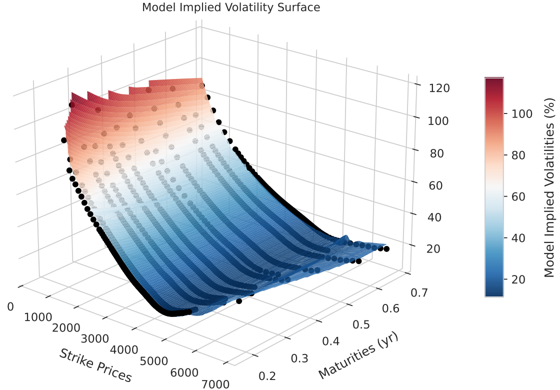
- The model provide a fairly good fit of the long expiries volatility slices.

Joint Calibration of Options: Model-Implied Volatility Surface (From 4 to 7 months)

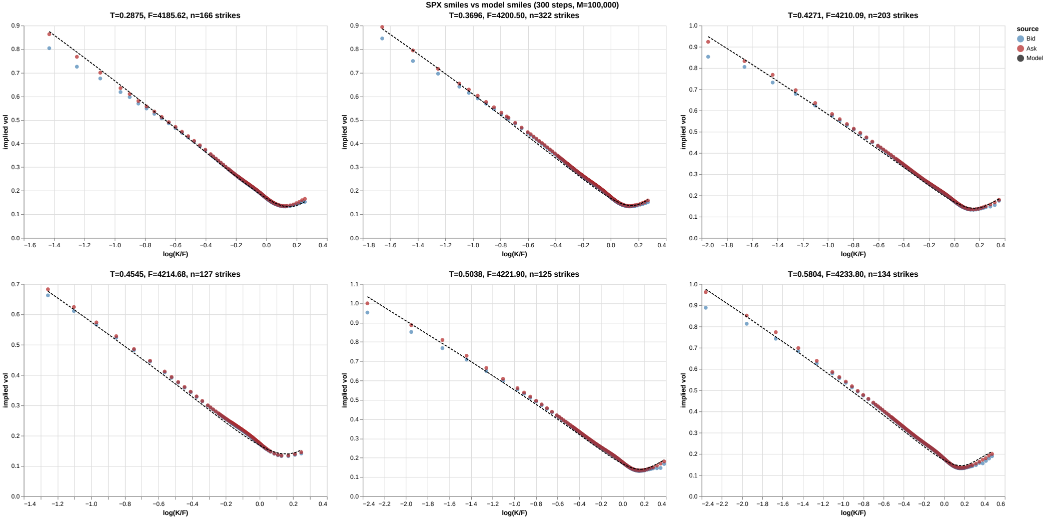
Expiries (in years)	$\alpha = H + \frac{1}{2}$	λ	ρ	c	m_0	κ_0	κ_2
0.2875, 0.3696,0.4271,0.4545,0.5038,0.5804	0.5929	1.3453	0.8985	0.5784	0.1139	0.0077	1.7289

Table: *Calibrated parameters for the volatility surface.*





Joint Calibration of Options: Model-Implied Volatility Surface



- The model adapt to each patterns of the differents expiries, producing a fairly good fit to the market data.

Joint Calibration of Options: Model-Implied Volatility Surface (out of samples), from 1 to 3 weeks

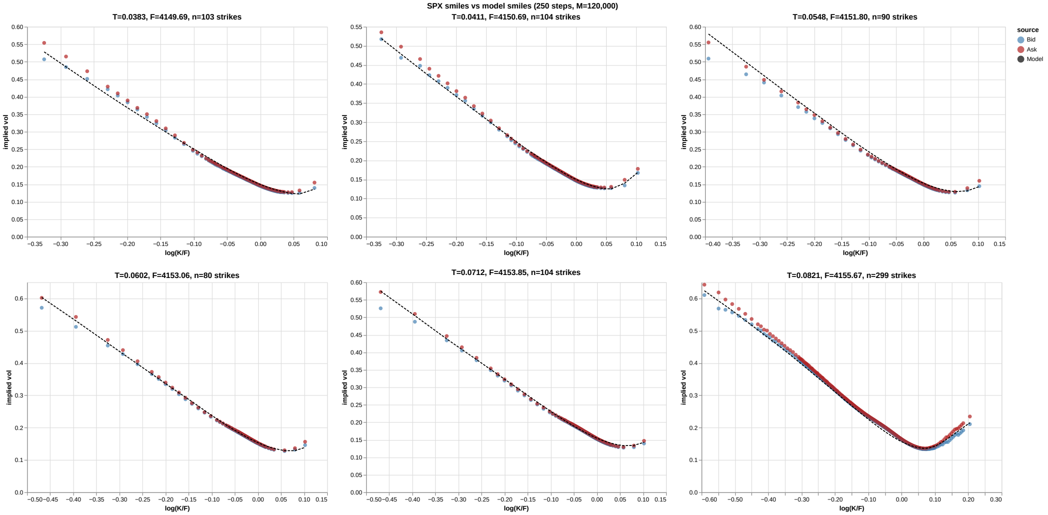


Figure: *Calibrated SPX Implied Volatility surface (Volatility smiles for maturities between one week and one month)*

Gnabeyeu, E. and Pagès, G. (2025).
On a stationarity theory for stochastic volterra integral equations with affine drift.
https://doi.org/10.48550/arXiv.2511.03474.

Gnabeyeu, E. and Pagès, G. (2026).
On path-dependent volterra integral equations: Strong well-posedness and stochastic numerics.
https://doi.org/10.48550/arXiv.2603.20996.

Gnabeyeu, E., Pagès, G., and Rosenbaum, M. (2025).
On inhomogeneous affine volterra processes: stationarity and applications to the volterra heston model.
https://doi.org/10.48550/arXiv.2512.09590.

Gnabeyeu, E., Pagès, G., and Rosenbaum, M. (2026).
Fake stationary rough heston volatility: Microstructure-inspired foundations.
https://doi.org/10.48550/arXiv.2602.11032.

Pagès, G. (2024).
Volterra equations with affine drift: looking for stationarity. application to quadratic rough heston model.
forthcoming in *Bernoulli*.

Thanks For Your Attention! Questions ?