On a Stationarity Theory for Stochastic Volterra Integral Equation with affine drift: Finite-Time, Functional Weak Asymptotics and Applications.

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The *short/rough* and *long-term memory* of Volterra Equations
Centre de Recerca Matemàtica - Universitat de Barcelona.

Stochastic Volterra Integral Equation (Convolutive kernel)

We are interested in the stochastic Volterra integral equation:

$$X_t = X_0\phi(t) + \int_0^t K(t-s)(\mu(s) - \lambda X_s)ds + \int_0^t K(t-s)\sigma(s, X_s)dW_s, \quad X_0 \perp\!\!\!\perp W. \tag{1}$$

where $\mu : [0, T] \to \mathbb{R}$ is a bounded Borel function (hence having a well-defined finite Laplace transform on $\mathbb{R}_+$).

- $\lambda > 0$, $\sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ Lipstchiz continuous function,ϕ a deterministic continuous function.
- $(W_t)_{t\geq 0}$ is a standard Brownian motion, independent of X_0, both defined on a probability space $(\Omega, \mathcal{A}, P)$.
- Let $\mathcal{F}_t \supset \mathcal{F}_{t,X_0,W}$ be a filtration satisfying the usual conditions.
- K is a convolutive kernel, i.e. a kernel $K : \{(s,t) \in \mathbb{R}_+^2 : 0 \leq s < t\} \to \mathbb{R}_+$ satisfying

$$\forall s, t \geq 0, \ s < t, \quad K(s,t) = K(0, t-s) \tag{2}$$

Summary of Main Tools

1. **Resolvent of a Convolutive Kernel**: The λ-resolvent R_λ is defined as the unique solution – if it exists – to

$$\forall t \geq 0, \quad R_\lambda(t) + \lambda \int_0^t K(t-s)R_\lambda(s)\,ds = 1 \quad \Longrightarrow R_\lambda = \sum_{k\geq 0}(-1)^k \lambda^k (1 * K^{*k}) \tag{3}$$

It is used as a generalisation of the classical application of **Itô's Lemma:** to $e^{\lambda t}X_t$.

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Example (Focus on fractional kernels $K(t) = K_\alpha(t) = \frac{u^{\alpha-1}}{\Gamma(\alpha)} \mathbf{1}_{\mathbb{R}}(t)$)

- $K_\alpha * K_{\alpha'} = K_{\alpha+\alpha'}$ implies $t \geq 0$,

$$R_{\alpha,\lambda}(t) = \sum_{k\geq 0} (-1)^k \frac{\lambda^k t^{\alpha k}}{\Gamma(\alpha k + 1)} = E_\alpha(-\lambda t^\alpha), \quad \text{So that} \quad f_{\alpha,\lambda}(t) = -R'_{\alpha,\lambda}(t) = \lambda t^{\alpha-1} \sum_{k\geq 0} (-1)^k \lambda^k \frac{t^{\alpha k}}{\Gamma(\alpha(k+1))}.$$

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2. **Wiener-Hopf Equation:** Let $g : \mathbb{R}_+ \to \mathbb{R}$ be a Borel locally bounded function. Then, the Wiener-Hopf equation

$$\forall t \geq 0, \quad x(t) = g(t) - \lambda \int_0^t K(t-s) x(s)\, ds \tag{4}$$

has a unique solution given by

$$\forall t \geq 0, \quad x(t) = g(t) + \int_0^t R'_\lambda(t-s) g(s)\, ds = g(t) - \int_0^t f_\lambda(t-s) g(s)\, ds \quad \text{where} \quad f_\lambda = -R'_\lambda \tag{5}$$

Scaled Volterra Equation: Main Assumption

- **Standing assumption on the kernel K:**

Assumption (λ-resolvent R_λ of the kernel for every $\lambda > 0$:)

$$(\mathcal{K}) \quad \begin{cases} (i) & R_\lambda \quad \mathbb{R}^+ - \textit{differentiable } \lim_{t\to+\infty} R_\lambda(t) = a \in [0,1[, \quad R_\lambda(0) = 1, \\ (ii) & f_\lambda \in \mathcal{L}^2_{loc}(\mathbb{R}_+, Leb_1), \textit{Where} \quad f_\lambda = -R'_\lambda \textit{ for } t > 0, \\ (iii) & \phi \in \mathcal{L}^1_{\mathbb{R}_+}(Leb_1), \textit{ a continuous function satisfying } \quad \lim_{t\to\infty} \left(\phi(t) - (f_\lambda * \phi)(t)\right) = a \\ (iv) & \mu \textit{ is a } C^1\textit{-function such that } \|\mu\|_{sup} < \infty \textit{ and } \lim_{t\to+\infty} \mu(t) = \mu_\infty \in \mathbb{R}, \end{cases} \tag{6}$$

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- **As a consequence of Wiener-Hopf and stochastic/ordinary Fubini's theorems, equation (1) reads:**

$$X_t = X_0\left(\phi(t) - (f_\lambda * \phi)(t)\right) + \frac{1}{\lambda}\int_0^t f_\lambda(t-s)\mu(s)\,ds + \frac{1}{\lambda}\int_0^t f_\lambda(t-s)\sigma(s, X_s)\,dW_s. \tag{7}$$

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Looking for Stationarity !

- Either in the classical sense, where the distribution of the process is invariant under time shifts?
- or in a weaker sense ?

Stationarity of the Mean

By standard arguments (BDG inequality and $R_\lambda(t) = 1 - \int_0^t f_\lambda(t-s)\,ds$), the mean $\mathbb{E}[X_t]$ satisfies:

$$\forall t \geq 0, \quad (\phi(t) - 1)\mathbb{E}[X_0] + \int_0^t f_\lambda(t-s)\left(\frac{\mu(s)}{\lambda} - \phi(t)\mathbb{E}[X_0]\right) ds = 0. \tag{8}$$

Proposition ($\mathbb{E}(X_t)$ is constant if and only if:)

- $\phi(t) = 1$ *a.s.*, $\mu(t) = \mu_0 = \mu_\infty$ *a.s.*, $\mathbb{E}[X_0] = \frac{\mu_0}{\lambda}$.
  In this case, equation (7) *reads:*

$$\forall t \geq 0, \quad X_t = \frac{\mu_0}{\lambda} + \left(X_0 - \frac{\mu_0}{\lambda}\right) R_\lambda(t) + \frac{1}{\lambda}\int_0^t f_\lambda(t-s)\sigma(s, X_s)\,dW_s. \tag{9}$$

- $\phi(t) = 1 - \frac{\int_0^t f_\lambda(s)\left(\frac{\mu(t-s)}{\mu_\infty} - 1\right) ds}{\int_t^\infty f_\lambda(s)\,ds}$, $\mathbb{E}[X_0] = \frac{\mu_\infty}{\lambda}$.
  In this case:

$$\forall t \geq 0, \quad X_t = X_0 - \left(X_0 - \frac{\mu_\infty}{\lambda}\right)\int_0^t f_\lambda(t-s)\frac{\mu(s)}{\mu_\infty}\,ds + \frac{1}{\lambda}\int_0^t f_\lambda(t-s)\sigma(s, X_s)\,dW_s. \tag{10}$$

Stationarity of the Variance(time homogenous case):

- Assume $\sigma(t,x) := \sigma(x)$
- If $X_0 \in L^2(\mathbb{P})$: a necessary condition for stationarity is $(\mathbb{E}[X_t] = \frac{\mu_\infty}{\lambda})$ and

$$\forall t \geq 0, \quad \mathbb{E}[X_t] = \frac{\mu_\infty}{\lambda}, \quad Var(X_t) = v_0 \quad \text{and} \quad \mathbb{E}[\sigma^2(X_t)] = \bar{\sigma}^2, \quad \text{are constant.} \tag{11}$$

By standard arguments (Itô's isomorphism and Fubini's Theorem) and assuming constant mean, the variance $Var[X_t]$ satisfies:

$$\forall\, t \geq 0, \quad \mathrm{Var}(X_t) = \mathrm{Var}(X_0)\phi^2(t)R_\lambda^2(t) + \frac{1}{\lambda^2}(f_\lambda^2 * \bar{\sigma}^2)(t)$$

Theorem (Time homogenous diffusion coefficient σ. Let $\sigma(t,x) = \sigma(x)$ in equation (7))

Assume that $X_0 \in L^2(\mathbb{P})$ *with* $\mathbb{E}[X_0] = \frac{\mu_\infty}{\lambda}$*. Suppose the conditions* (11) *hold for all* $t \geq 0$*:*
Then, only two situations can occur: either

- $\bar{\sigma}^2 > 0$ *and the kernel* K *is constant so that* $(X_t)_{t\geq 0}$ *is a (Markov) Brownian diffusion process.*

or

- $\bar{\sigma}^2 = 0$ *and* $\forall t \geq 0, \quad X_t = \frac{\mu_\infty}{\lambda} \quad \mathbb{P} - a.s$

Morally speaking. . .

True Volterra equations have no stationary regime !

Then, why not fake stationary regimes? . . .

We investigate the case where: $\sigma(t,x) = \underbrace{\varsigma(t)}_{\text{stabilizer?!}} \times \sigma(x), \quad \varsigma(t), \sigma(x) > 0.$

Definition (Fake Stationary Regime of type I and II [2] G.Pagès 2024)

1. The process $(X_t)_{t\geq 0}$ starting from $X_0 \in L^2(\mathbb{P})$, exhibit a fake stationary regime of type I if:

$$\forall\, t \geq 0, \quad \mathbb{E}\, X_t = c^{ste}, \quad \mathrm{Var}(X_t) = c^{ste} \geq 0 \quad \text{and} \quad \bar{\sigma}^2(t) := \mathbb{E}\, \sigma^2(X_t) = c^{ste} := \bar{\sigma}_0^2 \geq 0. \tag{12}$$

2. The process $(X_t)_{t\geq 0}$ starting from $X_0 \in \bigcap_{p>0} L^p(\mathbb{P})$, exhibit a fake stationary regime of type II if $(X_t)_{t\geq 0}$ has the same marginal distribution, i.e., $X_t \overset{d}{=} X_0$ for every $t \geq 0$.

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Theorem (Time-Dependent Volatility σ. Let $\sigma(t,x) = \varsigma(t)\sigma(x)$ in equation (7))

Assume that $X_0 \in L^2(\mathbb{P})$ *with* $\mathbb{E}[X_0] = \frac{\mu_\infty}{\lambda}$. *Suppose the following conditions hold for all* $t \geq 0$*:*

$$\mathbb{E}[X_t] = \frac{\mu_\infty}{\lambda}, \quad Var(X_t) = Var(X_0) = v_0 \geq 0, \quad \textit{and} \quad \mathbb{E}[\sigma^2(X_t)] =: \bar{\sigma}^2(t) = \bar{\sigma}_0^2 \geq 0. \tag{13}$$

Then, a necessary condition for (13) *to hold is that* $(v_0, \bar{\sigma}_0^2, \varsigma(t))$ *must satisfy the following Master equation :*

$$(E_{\lambda,c}): \quad \forall\, t \geq 0, \quad c\lambda^2\left(1 - \phi^2(t) R_\lambda^2(t)\right) = (f_\lambda^2 * \varsigma^2)(t) \quad \textit{where} \quad c = \frac{v_0}{\bar{\sigma}_0^2} \quad \textit{so that} \quad \varsigma = \varsigma_{\lambda,c}. \tag{14}$$

Examples of fake stationary regimes I and II:

Let's consider a squared trinomial diffusion coefficient given below:

$$\sigma(x) = \sqrt{\kappa_0 + \kappa_1(x-a) + \kappa_2(x-b)^2} \quad \text{with} \quad \kappa_i \geq 0,\ i = 0, 2,\ \kappa_1^2 \leq 4\kappa_2(\kappa_0 + \kappa_1(b-a)). \tag{15}$$

Proposition (Fake stationary regimes (types I and II) and asymptotics: [3])

Consider a stabilized Volterra Equation with the diffusion coefficient σ given by (15) and let $\lambda > 0$, $\mu_0 \in \mathbb{R}$, where $\varsigma = \varsigma_{\lambda,c}$, assumed to be the unique continuous solution to Equation $(E_{\lambda,c})$ in (14) for some $c \in (0, \frac{1}{[\sigma]^2_{Lip}})$. Then, let $X_0 \in L^2(P)$ with

$\mathbb{E}[X_0] = \frac{\mu_\infty}{\lambda}$, *and* $Var(X_0) = v_0 = \frac{c\sigma^2(\frac{\mu_\infty}{\lambda})}{1-c\kappa_2}$:

1. **Case $\sigma(x) = \sigma$ is constant.** *Here, the solution $(X_t)_{t\geq 0}$ has a fake stationary regime of type I with mean $\frac{\mu_\infty}{\lambda}$ and variance v_0.*
    - *This represents a fake stationary regime of type II if $X_0 \sim \nu^* := \mathcal{N}\left(\frac{\mu_\infty}{\lambda}, v_0\right)$, so that $X_t \overset{d}{=} X_0 \quad \forall t \geq 0$.*
2. **Case where σ is not constant and not degenerated (i.e. with $\sigma(\frac{\mu_\infty}{\lambda}) \neq 0$).** *The solution $(X_t)_{t\geq 0}$ to (7) has a fake stationary regime of type I, i.e., for all $t \geq 0$,*

$$\forall t \geq 0, \quad \mathbb{E}[X_t] = \frac{\mu_\infty}{\lambda}, \quad Var(X_t) = v_0 = \frac{c\sigma^2(\frac{\mu_\infty}{\lambda})}{1-c\kappa_2}, \quad \text{and} \quad \mathbb{E}[\sigma^2(X_t)] = \bar{\sigma}_0^2 = \frac{\sigma^2(\frac{\mu_\infty}{\lambda})}{1-c\kappa_2}.$$

Moreover, if $a = 0$, for any starting value $X_0 \in L^2(P)$, the process X mixes i.e. $\mathbb{E}[X_t] \to \frac{\mu_\infty}{\lambda}$ and $Var(X_t) \to v_0$ as $t \to +\infty$ (L^2 -confluence).

Long run behaviour: L^2-confluence

Proposition (L^2-confluence/Contraction Properties)

Assume $(\mathcal{K})$ *is in force, let* $\sigma(t,x) := \varsigma(t)\sigma(x)$ *where* $\varsigma = \varsigma_{\lambda,c}$ *is a non-negative, continuous and bounded solution to* $E_{\lambda,c}$ *for some fixed* $\lambda, c > 0$ *and* $\sigma : \mathbb{R} \to \mathbb{R}$ *is a (Locally) Lipschitz continuous function. Let* $X_0, X_0' \in L^2(\mathbb{P})$*, we consider the solutions to Volterra equation (1) denoted* $(X_t)_{t\geq 0}$ *and* $(X_t')_{t\geq 0}$ *starting from* X_0 *and* X_0' *respectively.*
Set $\rho := c[\sigma]^2_{Lip}$*. Then, one has, for* $c \in \left(0, \frac{1}{[\sigma]^2_{Lip}}\right)$ *:*

(a) *There exists a continuous non-negative function* $\varphi_\infty^{\lambda,c,K,\phi} =: \varphi_\infty : \mathbb{R}^+ \to [0,1]$*, such that* $\varphi_\infty(0) = 1$, $\lim_{t\to+\infty} \varphi_\infty(t) = \frac{a^2}{1-\rho(1-a^2)}$*, only depending on* λ, c, ϕ*, and the kernel* K*, such that :*

$$\forall t \geq 0,\ \mathbb{E}\left(\left|X_t - X_t'\right|\right)^2 \leq \varphi_\infty(t)\mathbb{E}\left(\left|X_0 - X_0'\right|\right)^2.$$

(b) *This result can be written using the 2-Wasserstein distance between* X *and* X'*:*

$$\forall t \geq 0,\ W_2([X_t'],[X_t]) \leq \varphi_\infty(t)^{1/2} W_2([X_0'],[X_0]),$$

(c) *In particular, whenever* $a = 0$,

- *if* X *has a fake stationary regime of type I,* $\mathbb{E}X_t' \to \frac{\mu_\infty}{\lambda}$, $Var(X_t') \to v_0$ *as* $t \to +\infty$. *(Finite-dimensional* W_2*-convergence.)*
- *In case if* X *has a fake stationary regime of type II, its marginal distribution is unique.*

Long run behavior $\sigma(t,x) = \varsigma_{\lambda,c}(t)\sigma(x)$: functional weak asymptotics (1)

Theorem (Long run theorem-a)

- Let λ, $c > 0$. Assume $(\mathcal{K})$ is in force and f_λ satisfy

$$\exists \beta > 1 : \int_0^{+\infty} f_\lambda(t)^{2\beta}\, dt < +\infty \quad \text{for some } \beta > 1\,, \quad \exists \vartheta \in (0,1] : \max_{i=1,2} \Big[\int_0^{+\infty} |f_\lambda(u+\bar{\delta}) - f_\lambda(u)|^i\, du \Big]^{\frac{1}{i}} \le C\bar{\delta}^{\vartheta}.$$

- Assume σ Lipschitz, $(E_{\lambda,c})$ in force with a stabilizer $\varsigma_{\lambda,c}$ and $c[\sigma]_{\text{Lip}}^2 < 1$
- Moreover, for some $\delta > 0$, for any $p > 0$ and $T > 0$, $t \to x_0(t) = X_0\phi(t)$ is absolutely continuous,and satisfy

$$\mathbb{E}\Big(\sup_{t\in[0,T]} |x_0(t)|^p \Big) < +\infty, \quad \mathbb{E}\big[\, |x_0(t') - x_0(t)|^p \,\big] \le C_{T,p}\Big(1 + \mathbb{E}\big[\sup_{t\in[0,T]} |x_0(t)|^p \big]\Big)|t'-t|^{\delta p}.$$

(a) The family of shifted processes $(X_{t+u})_{u\ge 0}$ is $\mathcal{C}$-tight and uniformly square integrable as $t \to +\infty$. There exists a process X^∞ with a $\Big(\delta \wedge \vartheta \wedge \frac{\beta-1}{2\beta} - \frac{1}{p} - \eta\Big)$-Hölder pathwise continuous sample paths for sufficiently small $\eta > 0$ such that

$$(X_{t+u})_{t\ge 0} \Rightarrow (X_t^\infty)_{t\ge 0} \quad \text{weakly in } C(\mathbb{R}_+;\mathbb{R}) \text{ as } u \to \infty.$$

Long run behavior $\sigma(t,x) = \varsigma_{\lambda,c}(t)\sigma(x)$: functional weak asymptotics (2)

Theorem (Long run theorem b-c)

If $(X_t)_{t\geq 0}$ of the SVIE (1) has a fake stationary regime of type I, starting from a random variable $X_0 \in L^2(P)$ with mean $\frac{\mu_\infty}{\lambda}$ and variance v_0,

(b) **Functional weak long-run behavior.** For $t_1, t_2 \geq 0$, $t_1 \leq t_2$,

$$\mathrm{Cov}(X_{t+t_1}, X_{t+t_2}) \overset{t\to+\infty}{\longrightarrow} \bar{C}_{f_\lambda}(t_1,t_2) := a^2 v_0 + \frac{(1-a^2)v_0}{\int_0^{+\infty} f_\lambda^2(s)ds} \int_0^{+\infty} f_\lambda(t_2 - t_1 + u) f_\lambda(u)du. \tag{16}$$

Thus, under any limiting distribution P, on the canonical space $\Omega_0 := \mathcal{C}(\mathbb{R}_+, \mathbb{R})$, the canonical process

$$Y_t(\omega) = \omega(t),\ \ \omega \in \Omega_0 \text{ is a } (weak) L^2\text{-stationary process with mean } \frac{\mu_\infty}{\lambda} \text{ and covariance function } \bar{C}_{f_\lambda}(s,t),\ \ s,t \geq 0.$$

(c) **Stationary Gaussian Case.** If $\sigma(x) = \sigma > 0$ (constant) and X_0 has a Gaussian distribution, say $X_0 \sim \mathcal{N}\left(\frac{\mu_\infty}{\lambda}, v_0\right)$, then $(X_t)_{t\geq 0}$ satisfies

$$X_{t+\cdot} \xrightarrow{(C)} \Xi(f_\lambda) \quad \text{as} \quad t \to +\infty,$$

where $\Xi(f_\lambda)$ is the stationary Gaussian process with mean $\frac{\mu_\infty}{\lambda}$ and covariance function $\bar{C}_{f_\lambda}(\cdot)$.

Applications: α-fractional kernels $K(t) = K_\alpha(t) = \frac{u^{\alpha-1}}{\Gamma(\alpha)}\mathbf{1}_{\mathbb{R}}(t)$: $\alpha \in (\frac{1}{2}, \frac{3}{2}) \subset (0,2)$, $\lambda > 0$.

Remark (Fractional integration kernel: $K(t) = K_\alpha(t) = \frac{u^{\alpha-1}}{\Gamma(\alpha)}\mathbf{1}_{\mathbb{R}}(t), \quad \alpha > 0.$)

The λ-resolvent $R_{\alpha,\lambda}$ of the kernel K and its derivative $f_{\alpha,\lambda} := -R'_{\alpha,\lambda}$ defined on $(0,+\infty)$ satisfy the scaling property:
$R_{\alpha,1} = e_\alpha, R_{\alpha,\lambda} = R_{\alpha,1}(\lambda^{1/\alpha}\cdot)$, $f_{\alpha,\lambda} = -\lambda^{\frac{1}{\alpha}} e'_\alpha(\lambda^{\frac{1}{\alpha}}\cdot)$ *with* $e_\alpha(t) := E_\alpha(-t^\alpha), \quad t \geq 0$ *(alternate Mittag-Leffler function)*

The representation of $e_\alpha(t)$, is obtained by applying the Cauchy residue theorem using the Hankel path $H_\alpha(\varepsilon)$.

$$e_\alpha(t) = F_\alpha(t) + G_\alpha(t), \quad t \geq 0, \tag{17}$$

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- With inverse Laplace transform,

$$F_\alpha(t) := \frac{1}{2\pi i}\int_{H_\alpha(\varepsilon)} e^{zt}\frac{z^{\alpha-1}}{z^\alpha + 1}\,dz = \int_0^\infty e^{-tu}H_\alpha(u)\,du, \quad \text{with} \quad H_\alpha(u) = \frac{\sin(\alpha\pi)}{\pi}\frac{u^{\alpha-1}}{u^{2\alpha} + 2u^\alpha\cos(\pi\alpha) + 1}, u \in \mathbb{R}_+$$

Applications: α-fractional kernels $K(t) = K_\alpha(t) = \frac{u^{\alpha-1}}{\Gamma(\alpha)}\mathbf{1}_{\mathbb{R}}(t)$: $\alpha \in \left(\frac{1}{2}, \frac{3}{2}\right) \subset (0,2)$, $\lambda > 0$.

Remark (**Fractional integration kernel:** $K(t) = K_\alpha(t) = \frac{u^{\alpha-1}}{\Gamma(\alpha)}\mathbf{1}_{\mathbb{R}}(t), \quad \alpha > 0.$)

The λ-resolvent $R_{\alpha,\lambda}$ of the kernel K and its derivative $f_{\alpha,\lambda} := -R'_{\alpha,\lambda}$ defined on $(0,+\infty)$ satisfy the scaling property:

$R_{\alpha,1} = e_\alpha, R_{\alpha,\lambda} = R_{\alpha,1}(\lambda^{1/\alpha}\cdot), f_{\alpha,\lambda} = -\lambda^{\frac{1}{\alpha}} e'_\alpha(\lambda^{\frac{1}{\alpha}}\cdot)$ *with* $e_\alpha(t) := E_\alpha(-t^\alpha), \quad t \geq 0$ (*alternate Mittag-Leffler function*)

The representation of $e_\alpha(t)$, is obtained by applying the Cauchy residue theorem using the Hankel path $H_\alpha(\varepsilon)$.

$$e_\alpha(t) = F_\alpha(t) + G_\alpha(t), \quad t \geq 0, \tag{17}$$

- With inverse Laplace transform,

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- With Cauchy residue theorem,

$$G_\alpha(t) := \sum_k e^{z_k t}\,\mathrm{Res}\left[\frac{z^{\alpha-1}}{z^\alpha+1}\right]_{z_k} = \frac{1}{\alpha}\sum_k e^{z_k t} = \frac{2}{\alpha}e^{t\cos\left(\frac{\pi}{\alpha}\right)}\cos\left(t\sin\left(\frac{\pi}{\alpha}\right)\right), \quad \text{where} \quad z_k = \exp\left(i\frac{(2k+1)\pi}{\alpha}\right)$$

$$e_\alpha(t) = \int_0^{+\infty} e^{-tu}H_\alpha(u)\,du + \frac{2}{\alpha}e^{t\cos\left(\frac{\pi}{\alpha}\right)}\cos\left(t\sin\left(\frac{\pi}{\alpha}\right)\right), \quad 0 < \alpha < 2.$$

Applications: Case of α-fractional kernels $K(t) = K_\alpha(t) = \frac{u^{\alpha-1}}{\Gamma(\alpha)}\mathbf{1}_{\mathbb{R}}(t)$: $\alpha \in \left(\frac{1}{2}, \frac{3}{2}\right)$, $\lambda > 0$.

Proposition ($\lambda > 0$ and $\alpha \in (1,2)$, $K_\alpha * K_{\alpha'} = K_{\alpha+\alpha'}$. [4])

(*a*) *The λ-resolvent $R_{\alpha,\lambda} = R_{\alpha,1}(\lambda^{1/\alpha}\cdot)$ is $\mathcal{C}^\infty$ on $\mathbb{R}_+$*, $R_{\alpha,\lambda}(0) = 1$, $a = \lim_{t\to+\infty} R_{\alpha,\lambda}(t) = 0$, $\lim_{t\to+\infty} f_{\alpha,\lambda}(t) = 0$

$$f_{\alpha,\lambda}(t) := R'_{\alpha,\lambda}(t) = \lambda^{\frac{1}{\alpha}} \int_0^{+\infty} e^{-\lambda^{\frac{1}{\alpha}} tu} u H_\alpha(u) du - \frac{2}{\alpha} e^{t\lambda^{\frac{1}{\alpha}} \cos(\frac{\pi}{\alpha})} \cos\left[t\lambda^{\frac{1}{\alpha}} \sin\left(\frac{\pi}{\alpha}\right) - \frac{\pi}{\alpha}\right].$$

(*b*) *For $i \in \{1,2\}$ and $\alpha \in \left(\frac{1}{2}, \frac{3}{2}\right)$, $f_{\alpha,\lambda} \in \mathcal{L}^{2\beta}(\mathrm{Leb}_1)$ $\forall \beta > 0$ and for every $\vartheta \in \left(0, \alpha - \frac{\mathbf{1}_{i=2}}{i}\right)$, there exists a real constant $C_{\vartheta,\lambda} > 0$ such that:* $\forall\, \delta > 0$, $\quad \left[\int_0^{+\infty} \left(f_{\alpha,\lambda}(t+\delta) - f_{\alpha,\lambda}(t)\right)^i\right]^{1/i} \leq C_{\vartheta,\lambda}\delta^\vartheta$.

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Corollary (Let $\sigma(t, x) = \varsigma(t)\sigma(x)$ with $\varsigma = \varsigma_{\alpha,\lambda,c}$ [2] and [3])

Let σ be given by equation (15), $X_0 \in L^2(\mathbb{P})$ such that $\mathbb{E}[X_0] = \frac{\mu_\infty}{\lambda}$ and $Var(X_0) = v_0 = \frac{c\sigma^2(\frac{\mu_\infty}{\lambda})}{1 - c\kappa_2}$. Then,

1. *The solution $(X_t)_{t\geq 0}$ to the SVIE (7) starting from X_0 has a fake stationary regime of type I:*
2. *If $c \in \left(0, \frac{1}{[\sigma]^2_{Lip}}\right)$, for every $X'_0 \in L^2(P)$,the solution to (7) starting from X'_0 satisfies $\|X'_t - X_t\|_2 \to 0$ as $t \to +\infty$.*
3. *The family of shifted processes $X_{t+\cdot}$, $t \geq 0$, is C-tight as $t \to +\infty$, and its (functional) limiting distributions are all L^2-stationary processes.*

Applications: Computing the Stabilizer $\varsigma_{\alpha,\lambda,c}$ for the α-fractional kernels $K_\alpha(t) = \frac{u^{\alpha-1}}{\Gamma(\alpha)}\mathbf{1}_{\mathbb{R}}(t)$:

Total Variation (Tauberian theorem) on Laplace transforms of the Master Equation suggests to search $\varsigma^2(t)$ as an expansion of the form (*Exponential Power Series Ansatz*): *Set* $a_k = \frac{1}{\Gamma(\alpha k+1)}, b_k = \frac{1}{\Gamma(\alpha(k+1))}, \ k \geq 0.$

$$\varsigma^2_{\alpha,\lambda,c}(t) = c\lambda^{2-\frac{1}{\alpha}}\varsigma^2_\alpha(\lambda^{\frac{1}{\alpha}}t) \quad \textit{where} \quad \varsigma^2_\alpha(t) := 2\,t^{1-\alpha}\sum_{k\geq 0}(-1)^k c_k t^{\alpha k}. \tag{18}$$

with

$$c_0 = \frac{\Gamma(\alpha)^2}{\Gamma(2\alpha-1)\Gamma(2-\alpha)}\frac{\mu(0)}{\mu_\infty}, \quad \textit{and for every } k \geq 1, \ c_k \textit{ is defined inductively by:}$$

$$c_k = \frac{\Gamma(\alpha)^2 B(\alpha(k+1),\, 2(1-\alpha))}{\Gamma(2(1-\alpha))\,\Gamma(2\alpha-1)}\left[(a*b)_k - \alpha(k+1)\sum_{\ell=1}^{k} B(\alpha(\ell+2)-1,\, \alpha(k-\ell-1)+2)\,(b^{*2})_\ell\, c_{k-\ell}\right].$$

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Proposition (Existence and Properties of the function $\varsigma^2_{\alpha,\lambda,c}$ for $\alpha \in (\frac{1}{2},\frac{3}{2})$)

The convergence radius of the fractional power series (18) *that defines* $\zeta_{\alpha,\lambda,c}$ *is infinite and* $\zeta_{\alpha,\lambda,c}$ *is positive on* $(0,+\infty]$ *so that* $\zeta_{\alpha,\lambda,c}$ *is well-defined: The stabilizer* $\zeta^2_{\alpha,\lambda,c}$ *exists as an non-negative function, such that:*

1. $\lim_{t\to 0}\zeta_{\alpha,\rho,\lambda,c} = \begin{cases} 0 \textit{ if } \alpha \leq 1, \\ +\infty \textit{ if } \alpha > 1 \textit{ provided } \frac{\mu(0)}{\mu_\infty} > 0. \end{cases}$ *and* $\lim_{t\to+\infty}\zeta_{\alpha,\lambda,c}(t) = \frac{\sqrt{c(1-a^2)\lambda}}{\|f_{\alpha,\lambda}\|_{L^2(Leb_1)}}$.

Example I: On The Stabilized Long-Memory Quadratic Volterra dynamic, $\alpha \in (1, \frac{3}{2}) \subset (1,2)$

We consider a squared trinomial diffusion coefficient σ of the form (15) for a certain $\lambda > 0$, and let c such that $c[\sigma]_{Lip}^2 < 1$:

$$\sigma(x) = \sqrt{\kappa_0 + \kappa_1\,(x - \frac{\mu_0}{\lambda}) + \kappa_2\,(x - \frac{\mu_0}{\lambda})^2} \quad \text{with} \quad \kappa_i \geq 0,\ i = 0, 2,\ \kappa_1^2 \leq 4\kappa_2\kappa_0\,, \lambda > 0. \tag{19}$$

We introduce a semi-integrated Euler-Maruyama scheme on the time grid $t_k = t_k^n = \frac{kT}{n}, k = 0, \ldots, n$:

$$\overline{X}_{t_k} = \frac{\mu_0}{\lambda} + (X_0 - \frac{\mu_0}{\lambda})R_\lambda(t_k) + \sum_{\ell=1}^{k} \frac{1}{\lambda} \int_{t_{\ell-1}}^{t_\ell} f_\lambda(t_k - s)\, \varsigma_{H+\frac{1}{2},\lambda,c}(t_\ell)\sigma(\overline{X}_{t_{\ell-1}})dW_s$$

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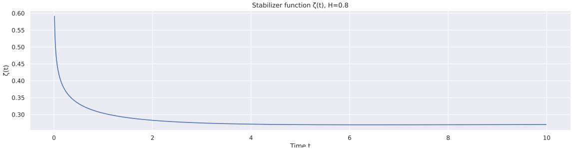

Figure: Graph of the stabilizer $t \to \varsigma_{\alpha,\lambda,c}(t)$ over time interval [0, T], T = 10 for a value of the Hurst esponent $H = 0.8$, $\lambda = 0.2$, c = 0.3.

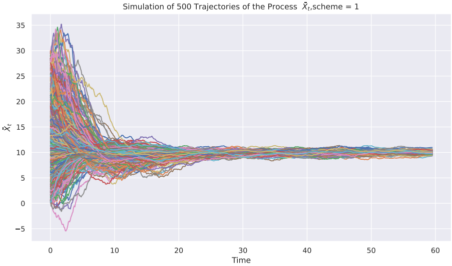

Figure: Confluence from a [0,30]-Uniform Distribution, T=60, $H = 0.8$, $\lambda = 0.2$, c = 0.36.

Example II: On The Stabilized Quadratic Rough Heston model

$$\frac{dS_t}{S_t} = \sigma(X_t)dW_t, \qquad \sigma(x) = \sqrt{\kappa_0 + \kappa_1\,(x - \tfrac{\mu_\infty}{\lambda}) + \kappa_2\,(x - \tfrac{\mu_\infty}{\lambda})^2} \quad \text{with} \quad \kappa_i \geq 0,\ i = 1, 2,\ \kappa_1^2 \leq 4\kappa_0\kappa_2.$$

where W is a BM, and μ_∞, λ are some positive constants, and X_t follows a quadratic Volterra stochastic process.

$$X_t = X_0 - \left(X_0 - \frac{\mu_\infty}{\lambda}\right)\int_0^t f_{\alpha,\lambda}(t-s)\frac{\mu(s)}{\mu_\infty}ds + \frac{1}{\lambda}\int_0^t f_{\alpha,\lambda}(t-s)\varsigma_{H+\frac{1}{2},\lambda,c}(s)\,\sigma(X_s)dW_s, \quad \varsigma_{\alpha,\lambda,c}(t),\ \sigma(x) \geq 0. \tag{20}$$

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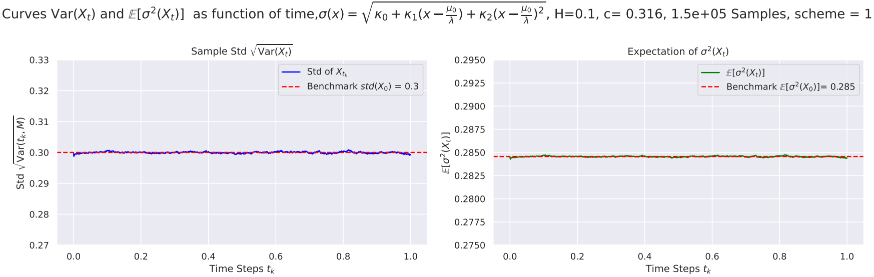
- **The Stabilized Quadratic Rough Heston volatility dynamic** ($V_t = \sigma^2(X_t)$ and $\alpha \in (\frac{1}{2}, 1)$): joint calibration of the S&P 500 and VIX smile, accounting for the so-called Zumbach effect.[2] and [3]



Curves Var(X_t) and $\mathbb{E}[\sigma^2(X_t)]$ as function of time,$\sigma(x) = \sqrt{\kappa_0 + \kappa_1(x - \frac{\mu_0}{\lambda}) + \kappa_2(x - \frac{\mu_0}{\lambda})^2}$, H=0.1, c= 0.316, 1.5e+05 Samples, scheme = 1

Figure: Graph of $t_k \mapsto \text{StdDev}(t_k, M)$ and $t_k \mapsto \mathbb{E}[\sigma^2(X_{t_k}, M)]$ over the time interval $[0, T]$, $T = 1$, $H = 0.1$, $\mu_0 = 2$, $\lambda = 0.2$, $v_0 = 0.09$ and $\text{StdDev}(X_0) = 0.3$. Number of steps: $n = 800$, Simulation size: $M = 150000$.

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Thanks For Your Attention!

Questions ?